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About the Author

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Precalculus

Chapter 1: Functions and Graphs

PDF Version

1.6: Inverse Relations

An inverse relation can be obtained by interchanging the variables x and y in a given relation. This means that, if a point (a, b) lies on the graph of a relation, then the point (b, a) lies on the graph of the inverse relation. That is, the domain of the relation becomes the range of the inverse, and vise-versa.

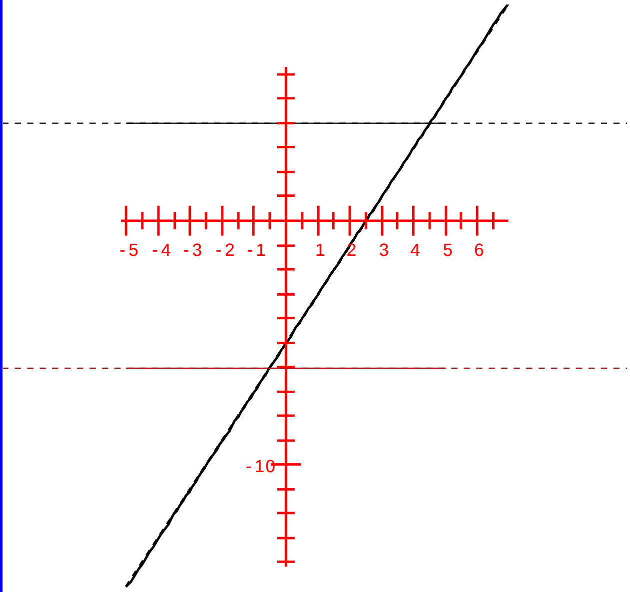
We know that a relation in which each value of x is mapped into a unique value of y is called a function. A function can be either many-to-one or one-to-one. While the graph of any function passes the vertical line test, the graph of a one-to-one function has unique property that each horizontal line intersects the graph in at most one point. That is, a one-to-one function passes both vertical line test and horizontal line test.

An inverse relation is a function if and only if the original relation is a one-to-one function. This is to say that inverse function exists if and only if the given function is one-to-one. Moreover, the inverse is also one-to-one.

Example 1:

Determine whether the function defined by $f(x) = 2x - 5$ is one-to-one.

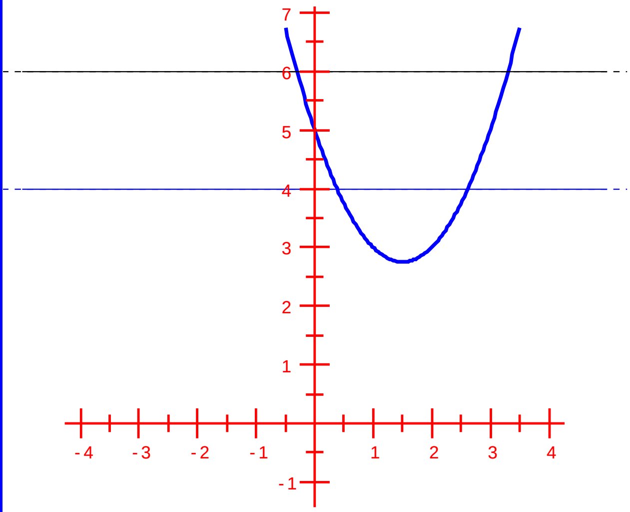
Solution: We see that, for each value of y, we get a unique value of x by substituting in the equation $y = 2x - 5$. So, this is a one-to-one function.



EXAMPLE 2:

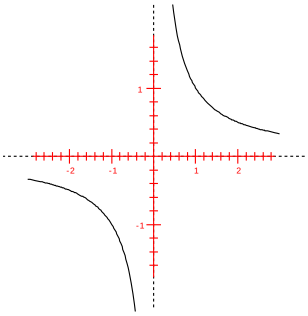
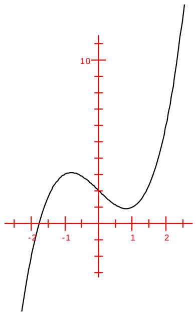
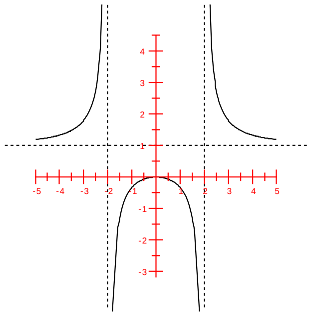
Determine whether the relation defined by $y = x^2 - 3x + 5$ is one-to-one.

Solution: We see that, solving for y gives to solutions in terms of x. So, this is not a one-to-one function.



EXAMPLE 3:

Determine whether each of the given graph is that of a one-to-one function:

(i)  (ii)  (iii) 

Solution: (i) Yes; (ii) No; (iii) Yes.

To find the inverse of a function algebraically, the following steps are used:

1. Rewrite the function in the form of $y = f(x)$
2. Interchange x and y and rewrite the expression
3. Solve for y

The domain of the inverse function should be determine with consideration of both the inverse function as well as the domain inherited from the given function.

EXAMPLE 4:

Find the inverse of the function $f(x) = 3x - 5$. Give the domain of both the function and its inverse.

Solution:

$$\begin{aligned} f(x) &= 3x - 5 \\ y &= 3x - 5 \\ x &= 3y - 5 \\ x + 5 &= 3y \\ \frac{x+5}{3} &= y \\ f^{-1}(x) &= \frac{x+5}{3} \end{aligned}$$

Domain of $f(x)$ and the domain of $f^{-1}(x)$ are is $(-\infty, \infty)$ as both are linear functions.

Example 5:

Find the inverse of the function $f(x) = \frac{x+1}{x}$ State the domain of the function, and the domain of the inverse.

Solution:

$$\begin{aligned}
f(x) &= \frac{x+1}{x} \\
y &= \frac{x+1}{x} \\
x &= \frac{y+1}{y} \\
xy &= y+1 \\
xy - y &= 1 \\
y(x-1) &= 1 \\
y &= \frac{1}{x-1} \\
f^{-1}(x) &= \frac{1}{x-1}
\end{aligned}$$

Domain of $f(x)$ is $(-\infty, 0) \cup (0, \infty)$; To determine the domain of $f^{-1}(x)$ is $(-\infty, 1) \cup (1, \infty)$.

Example 6:

Find the inverse of the function $f(x) = \sqrt{x+2}$. Determine the domain of the given function and its domain.

Solution:

$$f(x) = \sqrt{x+2}$$
$$y = \sqrt{x+2}$$
$$x = \sqrt{y+2}$$
$$x^2 = y+2$$
$$x^2 - 2 = y$$
$$f^{-1}(x) = x^2 - 2$$

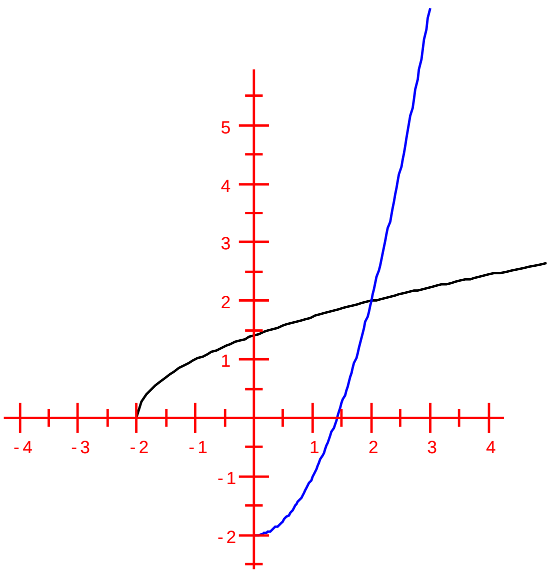
Domain of $f(x)$ is $[-2, \infty)$; $f^{-1}(x)$ is a quadratic function and,as it is, domain should be all real numbers. But, this inverse has an inherited/implied domain restriction of $x \geq 0$ as the range of $f(x)$ is all non-negative real numbers. This makes the domain of $f^{-1}(x)$ as $[0, \infty)$.

Graphs of inverse functions are reflection of each other on the identity line $y = x$. This property helps us to graph the inverse of a given graph.

Example 7:

Graph the function $f(x) = \sqrt{x-3}$ and its inverse.

Solution: In the graph shown below, graph of the given function is shown in black and the graph of the inverse is shown in blue.



Another unique property of inverse function is that their composite function is always the identity function. This is the reason their graphs are reflections on the identity line. That is, $(f \circ g)(x) = (g \circ f)(x) = x$. This property can be used to verify whether a given pair of functions are inverses or not.

EXAMPLE 8:

Check whether the functions $f(x) = \frac{x}{x+1}$ and $g(x) = \frac{x}{1-x}$ are inverses or not.

Solution:

$$\begin{aligned}
(f \circ g)(x) &= f[g(x)] \\
&= f\left(\frac{x}{1-x}\right) \\
&= \frac{\frac{x}{1-x}}{\frac{x+1-x}{1-x}} \\
&= \frac{x}{1-x} * \frac{1-x}{1} \\
&= x
\end{aligned}$$

$$\begin{aligned}
(g \circ f)(x) &= g[f(x)] \\
&= g\left(\frac{x}{x+1}\right) \\
&= \frac{\frac{x}{x+1}}{\frac{x+1-x}{x+1}} \\
&= \frac{x}{x+1} * \frac{x+1}{1} \\
&= x \\
&= \frac{x}{x+1} * \frac{x+1}{1} \\
&= x
\end{aligned}$$

Yes. $f(x)$ and $g(x)$ are inverse functions.

Example 9:

Verify algebraically that the functions $f(x) = x^3 + 1$ and $g(x) = \sqrt[3]{x-1}$ are inverse functions.

Solution:

$$\begin{aligned}(f \circ g)(x) &= f[g(x)] \\ &= f(\sqrt[3]{x-1}) \\ &= (\sqrt[3]{x-1})^3 + 1 \\ &= x - 1 + 1 \qquad = x\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g[f(x)] \\ &= \sqrt[3]{x^3 + 1 - 1} \\ &= \sqrt[3]{x^3} \\ &= x\end{aligned}$$

Yes. The two functions are inverses.

Practice Problems

For problems #1 - 5, check algebraically whether the given relation is a function, and if it is, whether it is one-to-one.

(1) f(x) = {(1,3), (-1,3), (2,-5), (3,0), (-1,0)}

It is a function as each x- coordinate is mapped to exactly one y- coordinate. But it is not one-to-one as same y-coordinate corresponds to more than one x- coordinate.

(2) $f(x) = \frac{4x+3}{3y+2}$

Yes.

(3) $f(x) = \ln x + 2$

Yes.

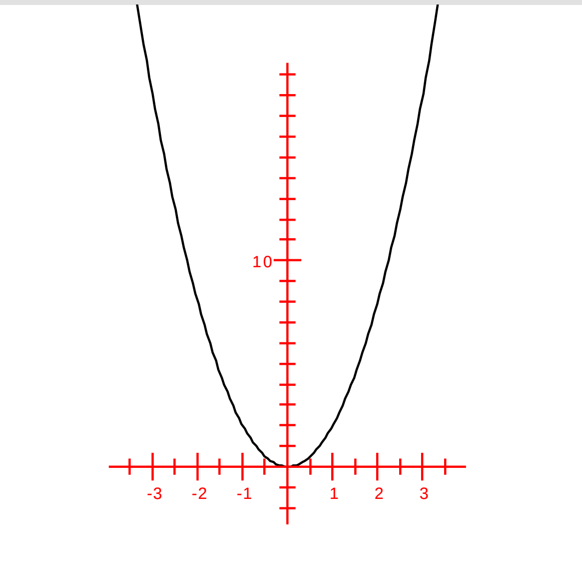
(4) $f(x) = x^2 - 2x + 5$

No.

(5) $f(x) = x^3 + 4x^2 - 2x - 3$

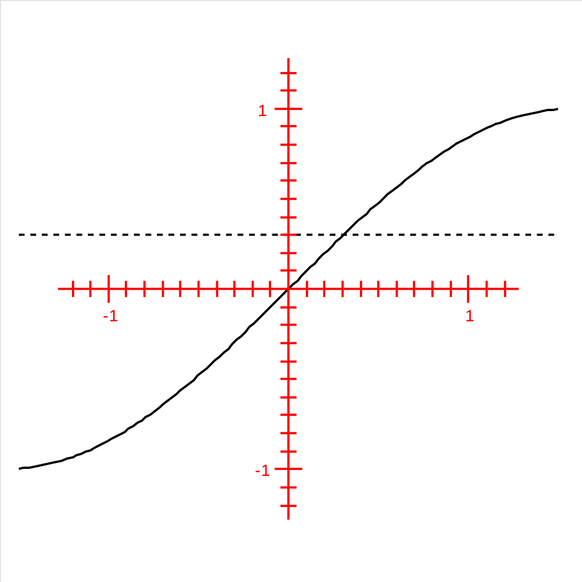
No

(6) Determine if the given graph is that of a one-to-one function.

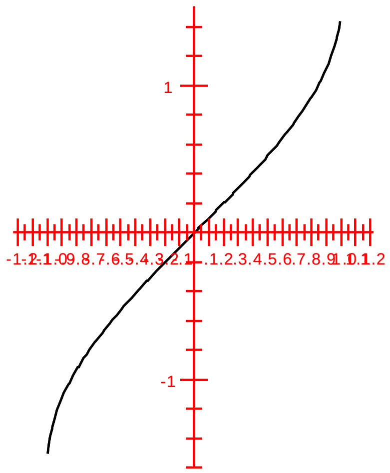


No.

(7) Determine whether the given graph is that of a one-to-one fur



Yes. Inverse graph is shown below:



For problems #8 - 15, find the inverse of the given function. also find the domain of the function and domain of its inverse.

(8) $f(x) = \frac{2x-3}{x+1}$

$f^{-1}(x) = \frac{x+3}{2-x}$. Domain of $f(x)$ is $(-\infty, -1) \cup (-1, \infty)$. Domain of the inverse is $(-\infty, 2) \cup (2, \infty)$

(9) $g(x) = x^3 + 5$

$g^{-1}(x) = \sqrt[3]{x-5}$. Domain of $g(x)$ is (∞, ∞). Domain of its inverse is (∞, ∞)

(10) $f(x) = \sqrt{x-3}$

$f^{-1}(x) = x^2 + 3$. Domain of $g(x)$ is $[3, \infty)$. Domain of its inverse is $[0, \infty)$

(11) $h(x) = \frac{x+1}{2}$

$h^{-1}(x) = 2x - 1$. Domain of $h(x)$ is $(-\infty, \infty)$. Domain of its inverse is also $(-\infty, \infty)$

(12)$p(x) = \frac{1}{x-1}$

$p^{-1}(x) = \frac{1+x}{x}$. Domain of $p(x)$ is $(-\infty, 1) \cup (1, \infty)$. Domain of its inverse is $(-\infty, 0) \cup (0, \infty)$

(13) $f(x) = \frac{2}{\sqrt{x+3}}$

$f^{-1}(x) = \frac{4-3x^2}{x^2}$. Domain of $f(x)$ is $(-3, \infty)$. Domain of its inverse is $(-\infty, 0) \cup (0, \infty)$

(14) $r(x) = x^2 - 4, \quad x > 0$

$r^{-1}(x) = \sqrt{x+4}$. Domain of $r(x)$ is $((0, \infty)$. Domain of its inverse if $(-4, \infty)$

(15) $f(x) = \frac{4}{x^2+1}, \quad x \geq 0$

$f^{-1}(x) = \frac{\sqrt{x-x}}{\sqrt{x}}$. Domain of $f(x)$ is $[0, \infty)$. Domain of its inverse is $(0, 4]$

(16) Show that the function $f(x) = x^3 + 3x + 2$ is one-to-one, us

Consider the function $g(x) = x^3 + 3x$. This is one-to-one as $g(-x) = -g(x)$. As $f(x)$ is a vertical shift of (x) up by 2 units, $f(x)$ is also one-to-one.

(17) Find the inverse of the function $f(x) = x^2 - 5, \quad x \geq 3$. Hen

$f^{-1}(x) = \sqrt{x+3}$; $f^{-1}(1) = 2$

(18) Find the inverse of $g(x) = 2(x+1)^2 - 1$ and hence evaluate

$g^{-1}(x) = \sqrt{\frac{x+1}{2}} - 1$; $g^{-1}(1) = 0$

(19) Find the inverse of $f(x) = \frac{x-1}{2} + 5$ and hence evaluate $f^{-1}($

$f^{-1}(x) = 2x - 9$; $f^{-1}(2) = -1$

(20) Find the inverse of $P(x) = 2 - \sqrt[3]{x+1}$ and hence evaluate I

$P^{-1}(x) = (2 - x)^3 - 1$; $P^{-1}(0) = 7$