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About the Author

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PRECALCULUS

Chapter 1: Functions and Graphs

PDF Version

1.5: Graphical Transformations

We can obtain more functions by adding, subtracting, multiplying, or dividing the basic functions by an integer. The graphs of these functions have shapes and sizes identical to the graph of parent (basic) functions. This process of building functions from the basic functions is called **Transformation** .

There are two types of transformations. **Rigid transformations** are those which leave the graph unchanged except the position. Translations and reflections are examples of such transformations.

Non-rigid transformations are those that distort the shape of the graph. Stretches and shrinks are examples of such transformations. We shall discuss these transformations in detail below.

Translations:
A translation of the graph of $y = f(x)$ is the shifting of the graph horizontally or vertically.
Let c be any positive number. Then, the following transformations result in translations of the graph of $y = f(x)$:

1. $y = f(x+c)$ will produce a translation to the left by c units.
2. $y = f(x-c)$ will produce a translation to the right by c units.
3. $y = f(x)+c$ will produce a translation up by c units.
4. $y = f(x)-c$ will produce a translation down by c units.

EXAMPLE 1:

Describe how the graph of $y = x^2$ can be transformed to the graph of the following functions:

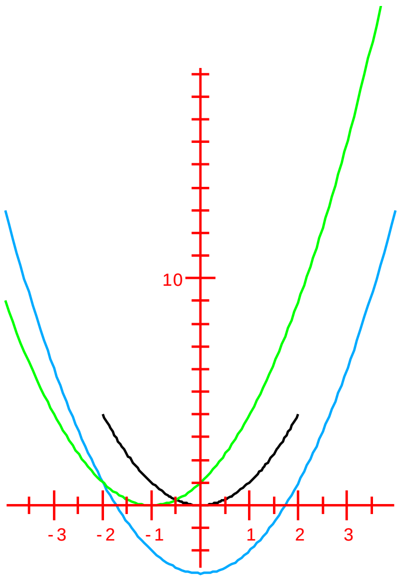
(a) $y = x^2 - 3$

(b) $y = (x+1)^2$

Sketch the graphs of these functions, along with the basic function.

Solution:(a) $x^2 - 3$ translates the basic function vertically down by 3 units.

(b) $y = (x+1)^2$ translates the basic function horizontally 1 unit to the left. In the graph shown below the black curve is the graph of the basic function $y = x^2$.The blue curve is the vertical translation of the basic function 3 unit below. This is the solution of (a). The green curve is the horizontal translation of the basic function, 1 unit to the left.



EXAMPLE 2:

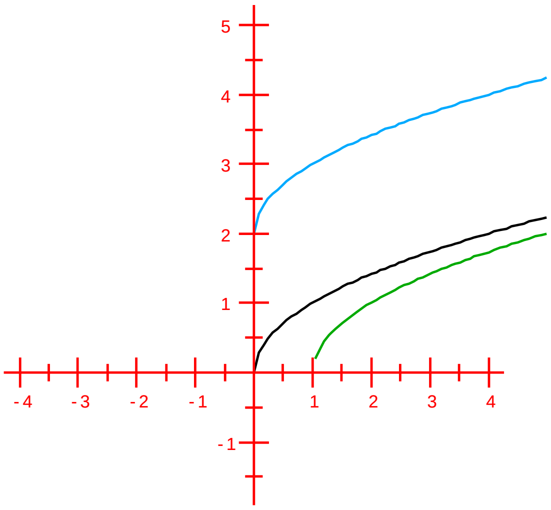
Describe how the graphs of the following functions can be obtained from a basic function:

(a) $y = \sqrt{x} + 2.$

(b) $y = \sqrt{x - 1}.$

Solution: (a) Basic function $y = |x|$ (black curve in the graph below) is vertically shifted 2 units above. (Blue curve in the graph below.)

(b) Basic function $y=|x| is horizontally shifted 3 units to the right.(Green curve in the graph below.)

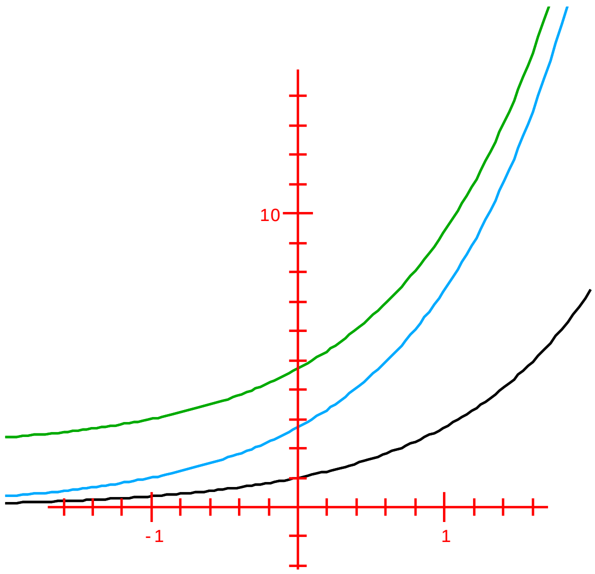


When both vertical and horizontal shifts are combined in a function, we do the transformations one by one.

Example 3:

Describe how you will obtain the graph of the function $y = e^{x+1} - 2$.

Solution: The graph of the basic function $y = e^x$ is first moved 1 unit to the left, and then the resulting graph is moved 2 units up.



Reflection:

Reflections are in a sense mirror images. Given a function $y = f(x)$, reflection of this function on the x-axis is the graph of the function $y = -f(x)$ and the reflection on the y-axis is the graph of the function $y = f(-x)$.

Example 4:

Given a function $y = x^3 - 3x^2$, obtain the equation of the reflection of this function on the x-axis and on the y axis. Given the graph of the function $y = x^3 - 3x^2$, sketch the graph each of these reflections.

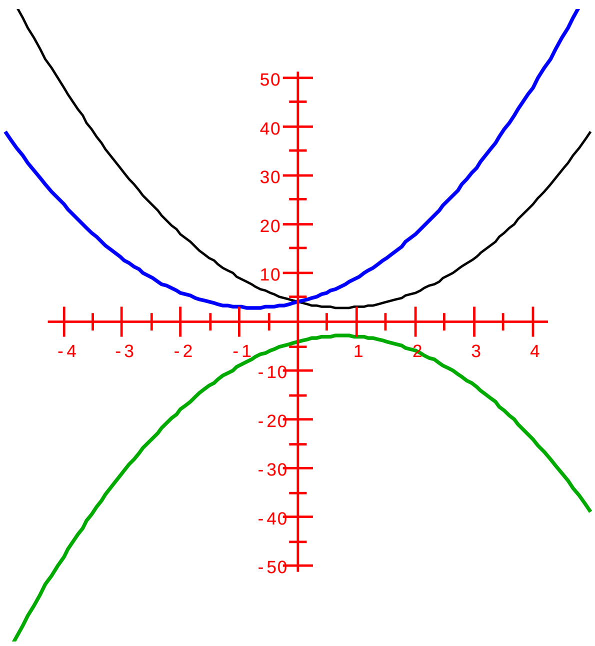
Solution: Equation of the reflection on y-axis is

$$\begin{aligned} f(-x) &= (-x)^3 - 3(-x)^2 \\ &= -x^3 - 3x^2 \end{aligned}$$

Equation of the reflection on x-axis is

$$\begin{aligned} -f(x) &= -(x^3 - 3x^2) \\ &= -x^3 + 3x^2 \end{aligned}$$

The following graph shows the graphs of the function $y = x^3 - 3x^2$ in black color, its reflection on x-axis in green color, and the reflection on y-axis in blue color.



EXAMPLE 5:

Given a function $f(x) = \frac{2x+3}{x-1}$, find the equations of the reflections on x-axis and y-axis.

Solution:Equation of the reflection on x-axis:

$$\begin{aligned} -f(x) &= -\frac{2x+3}{x-1} \\ &= \frac{2x+3}{1-x} \end{aligned}$$

Equation of the reflection on y-axis:

$$\begin{aligned} f(-x) &= \frac{2(-x)+3}{(-x)-1} \\ &= \frac{3-2x}{-x-1} \qquad = \frac{2x-3}{x+1} \end{aligned}$$

.

An even function has the property that $f(-x) = f(x)$.Therefore, an even function is its own reflection on the y-axis. That is to say that it is symmetric with respect to the y-axis.

Stretches and Shrinks

Generally, the graph of a function is distorted when we replace x by $\frac{x}{c}$ or y by $\frac{y}{c}$, where c is a constant. This transformation leads to stretches and shrinks. We can put it more conveniently as below:
Let c be a positive real number. Then, the following transformations result in stretches and shrinks of of the graph of $y = f(x)$:

1. $y = f\left(\frac{x}{c}\right)$ results in a horizontal stretch by a scale factor of c is $c > 1$ and a horizontal shrink by a scale factor of c if $c < 1$.
2. $y = c \times f(x)$ results in a vertical stretch by a scale factor of c if $c > 1$ and a vertical shrink by a scale factor of c if $c < 1$.

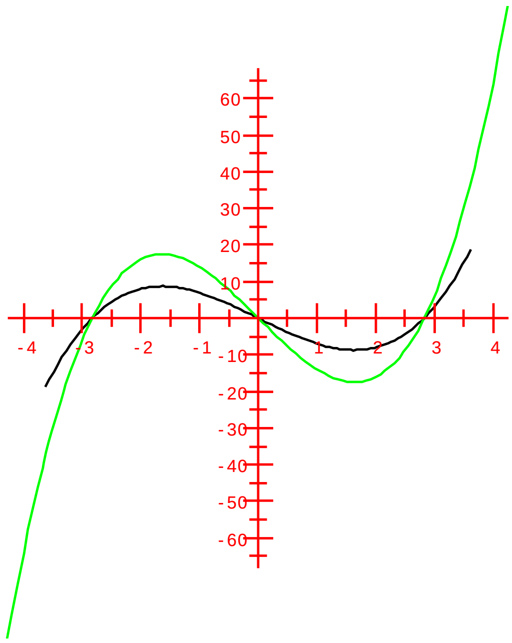
EXAMPLE 6:

Given a function $f(x) = x^3 - 8x$,
(a) Find the equation and graph the function resulting from a vertical stretch by a scale factor of 2.
(b) Find the equation and graph the function resulting from a horizontal shrink by a scale factor of $\frac{1}{2}$

Solution:(a) Equation:

$$y = 2 \times f(x)$$
$$y = 2(x^3 - 8x)$$
$$y = 2x^3 - 16x$$

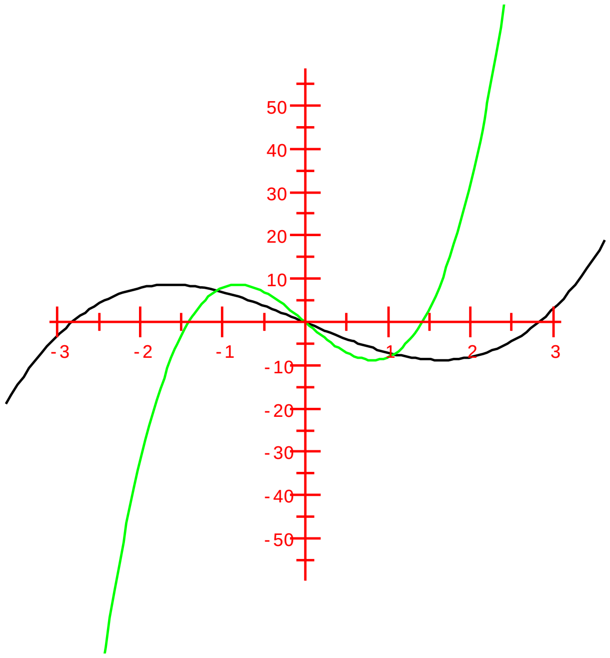
The graph of both original (black) and the resulting (green) equations are shown below.



(b) Equation:

$$y = f\left(\frac{x}{\frac{1}{2}}\right)$$
$$y = f(2x)$$
$$y = (2x)^3 - 8(2x)$$
$$y = 8x^3 - 16x$$

. The graph of both original (black) and the resulting (green equations are shown below.



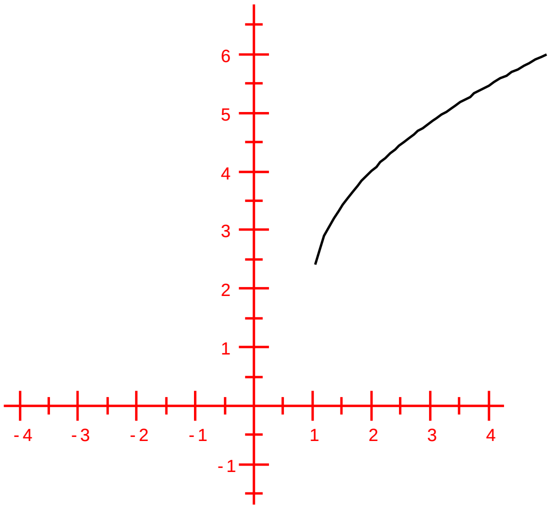
Sometimes, graphing an equation may require combination of two or more of the transformations. In that case, the order in which we perform the transformation matters and should be taken into consideration.

Example 7:

The graph of the equation $y = \sqrt{x}$ undergoes the following transformations: (i) A vertical shift of 2 units; (ii) a horizontal shift of 1 unit; and (iii) a vertical stretch by a scale factor of 2. Find the resulting equation and graph it.

Solution:Given equation and the transformations in order are as follows:

$$f(x) = \sqrt{x}$$
$$f(x) + 2 = \sqrt{x} + 2$$
$$f(x-1) + 2 = \sqrt{x-1} + 2$$
$$2[f(x-1) + 2] = 2\sqrt{x-1} + 2$$

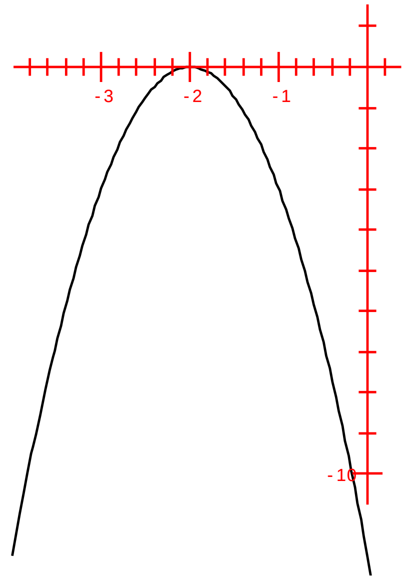


EXAMPLE 8:

Find the resulting equation if the graph of the function $f(x) = x^2$ undergoes the following transformations: (i) A horizontal translation of -2 units, (ii) A reflection on the y-axis, and (iii) A vertical stretch of a scale factor 3. Also graph the resulting equation.

Solution: Given equation and the transformations in order as as follows:

$$\begin{aligned} f(x) &= x^2 \\ f(x+2) &= (x+2)^2 \\ -f(x+2) &= -(x+2)^2 \\ -3f(x+2) &= -3(x+2)^2 \end{aligned}$$



Example 9:

Explain the transformations required to obtain the graph of $y = 2 + 3|x+1|$ in order,and then graph it.

Solution: Basic function:

$$f(x) = |x|.$$

Horizontal shift of -1 units:

$$f(x+1) = |x+1|$$

Vertical stretch of scale factor 3:

$$3f(x+1) = 3|x+3|$$

Vertical shift of 2 units:

$$2 + 3f(x+1) = 2 + 3|x+1|$$

Example 10:

Explain the transformations required to obtain the graph of $y = -(x-2)^2 + 1$ and graph it.

Solution: Basic function:

$$f(x) = x^2$$

Horizontal shift of 2 units:

$$f(x-2) = (x-2)^2$$

Reflection on x-axis:

$$-f(x-2) = -(x-2)^2$$

Vertical shift of 1 unit:

$$-f(x-2)+1 = -(x-2)^2+1$$

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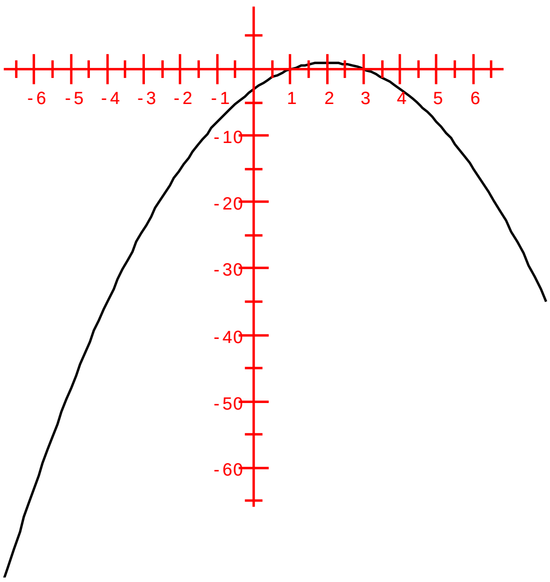


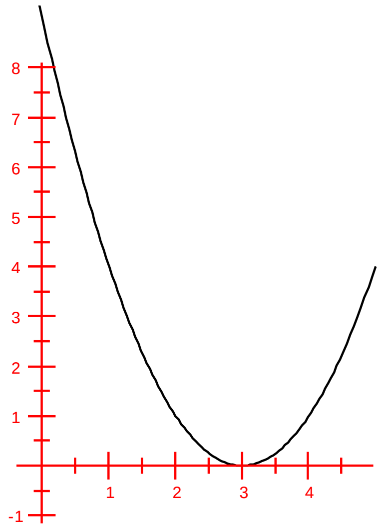
Practice Problems

For problems # 1 - 10, describe how the graph of the given function can be obtained by transformation of the basic graph.

State the basic graph, and also graph the resulting function.

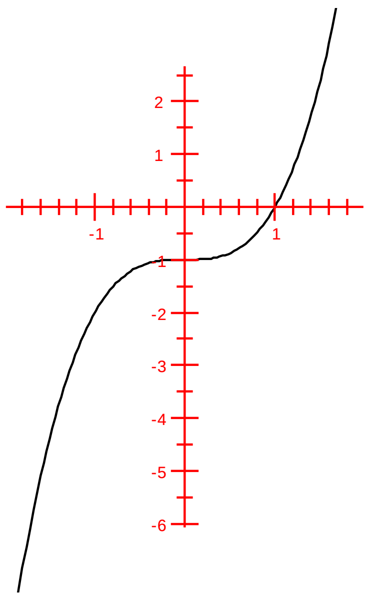
(1) $(x-3)^2$

Basic function: $f(x)=x^2$; Horizontal translation by 3 units: $f(x-3)=(x-3)^2$



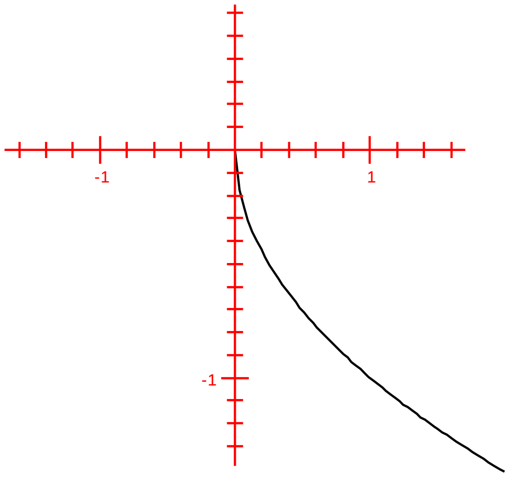
(2) $f(x)=x^3-1$

Basic function: $f(x)=x^3$; Vertical translation of -1 units: $f(x)-1=x^3-1$.



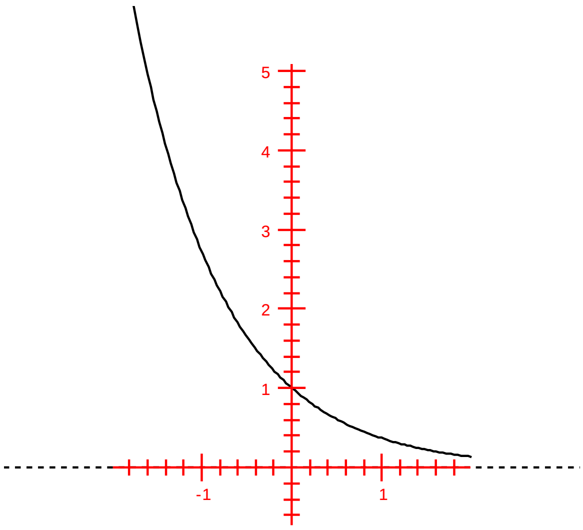
(3) $y = -\sqrt{x}$

Basic function: $f(x) = \sqrt{x}$; Reflection on x-axis: $-f(x) = -\sqrt{x}$.



(4) $f(x) = e^{-x}$

Basic function: $f(x) = e^{x}$; reflection on y- axis: $f(-x) = e^{-x}$



(5) $y = \frac{1}{2}x^3$

Basic function: $f(x) = x^3$; Vertical shrink by a scale factor of $\frac{1}{2}$

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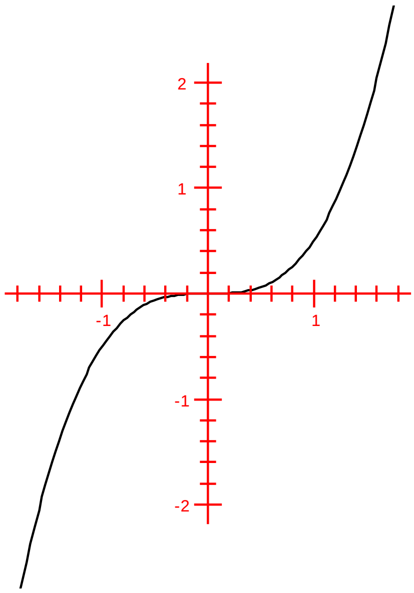
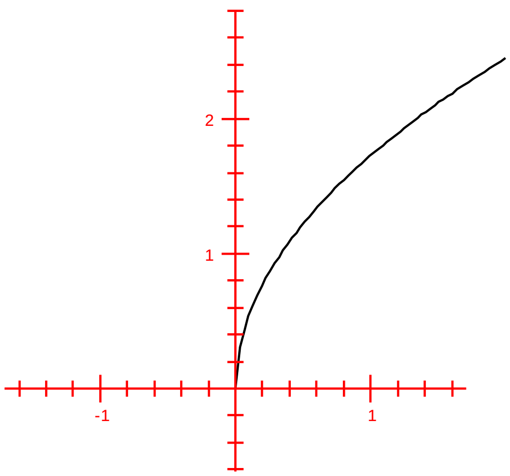


(6) $y = \sqrt{3x}$

Basic function: $f(x) = \sqrt{x}$; Horizontal shrink by a scale factor of $\frac{1}{3}$: $f(3x) = \sqrt{3x}$.

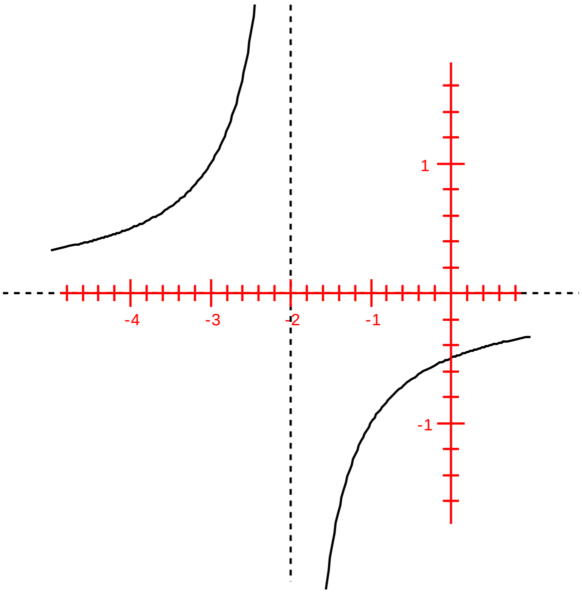


(7) $y = -|x - 2| + 1$

Basic function: $f(x) - |x|$; Horizontal shift of 2 units to the right: $f(x-2) = |x-2|$; Reflection on x-axis: $-f(x-2) = -|x-2|$; Vertical shift of 1 unit up: $-f(x-2)+1 = -|x-2|+1$.

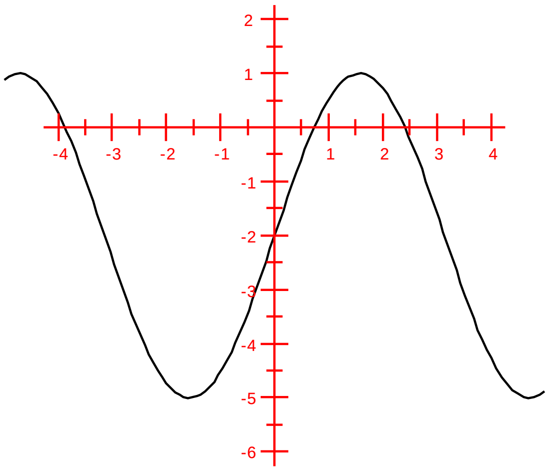
(8) $-\frac{1}{x+2}$

Basic function: $f(x) = \frac{1}{x}$; Horizontal shift to the left by 2 units: $f(x+2) = \frac{1}{x+2}$; Reflection on x-axis: $-f(x+2) = -\frac{1}{x+2}$



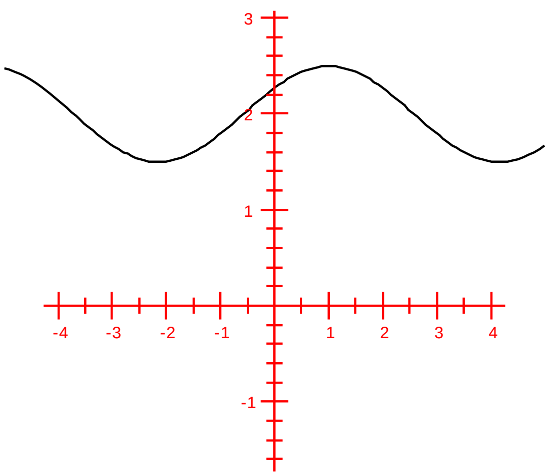
(9) $y = 3\sin x - 2$

Basic function: $f(x) = \sin x$; Vertical stretch by a scale factor of 3: $3f(x) = 3\sin x$; Vertical shift down by 2 units: $3f(x) - 2 = 3\sin x - 2$.



(10) $y = \frac{1}{2}\cos x - 1 + 2$

Basic function: $f(x) = \cos x$; Horizontal shift to the right by 1 unit: $f(x-1) = \cos x - 1$; Vertical shrink by a scale factor of $\frac{1}{2}$: $\frac{1}{2}f(x-1) = \frac{1}{2}\cos x - 1$; Vertical shift up by 2 units: $\frac{1}{2}f(x-1) + 2 = \frac{1}{2}\cos x - 1 + 2$



For problems #11 - 15, describe how to obtain the graph of the given function by a series of transformations, in order.

(11) $-3f(x+2)+1$

Horizontal shift to the left by 2 units: $f(x+2)$; Reflection on the x-axis: $-f(x+2)$; Vertical shift up by 1 unit: $-f(x+2)-1$.

(12) $f(2x)+3$

Horizontal shrink by a scale factor of $\frac{1}{2}$: $f(2x)$; Vertical shift up by 3 units: $f(2x)+3$

(13) $f(2-x)-1$

Reflection on y-axis: $f(-x)$; Horizontal shift to the left by 2 units: $f(2-x)$; Vertical shift down by 1 unit: $f(2-x)-1$

(14) $2-f(1-\frac{x}{3})$

Horizontal stretch by a scale factor of 3 units: $f(\frac{x}{3})$; Reflection on y-axis: $f(-\frac{x}{3})$; Horizontal shift to the left by 1 unit: $f(1-\frac{x}{3})$; Reflection on x-axis: $-f(1-\frac{x}{3})$; Vertical shift up by 2 units: $2-f(1-\frac{x}{3})$.

(15) $3f(x+1)-5$

Horizontal shift to the left by 1 unit: $f(x+1)$; Vertical stretch by a scale factor of 3: $3f(x+1)$; Vertical shift down by 5 units: $3f(x+1)-5$.

In problems #16 - 20, a graph $g(x)$ is obtained from the graph of $f(x)$ by a sequence of transformations indicated. Write an equation for the graph of $g(x)$.

(16) $f(x) = x^3$; a vertical stretch by a scale factor of 2, then a sh

$g(x) = 2(x+4)^3$

(17) $f(x) = |x|$; a reflection on x-axis; horizontal shrink by a scal

$g(x) = -|2x+5| + 4$

(18) $f(x) = \ln x$; horizontal shift by 5 units; vertical stretch by 3

$g(x) = 3 - \ln(x-5)$

(19) $f(x) = \sin x$; horizontal shrink by a scale factor of $\frac{1}{3}$; horizo

$g(x) = \sin(3x-2) - 4$

(20) $f(x) = \sqrt{x}$; horizontal translation to the right by 4 units; ver

$g(x) = 3\sqrt{x-4}$