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PRECALCULUS

Chapter 1: Functions and Graphs

PDF Version

1.3: Operations on Functions.

Given two functions $y = f(x)$ and $y = g(x)$, we can evaluate any expression involving addition, subtraction, multiplication, division, or composition of functions. Rules for doing this are stated below:

1. $(f+g)(x) = f(x) + g(x)$
2. $(f-g)(x) = f(x) - g(x)$
3. $(fg)(x) = [f(x)][g(x)]$
4. $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$
5. $(f \circ g)(x) = f[g(x)]$. This operation is called composition of two functions. The resulting function $f \circ g = f[g(x)]$ is the composite function of $f(x)$ and $g(x)$.

EXAMPLE 1:

Let $f(x) = 3x + 2$ and $g(x) = \frac{3x-2}{x}$, find and simplify the following:

(a) $(fg)(x)$

(b) $(f-g)(x)$

Solution: (a)

$$\begin{aligned}(fg)(x) &= (3x+2)\left(\frac{3x-2}{x}\right)\\ &= \frac{(3x+2)(3x-2)}{x}\\ &= \frac{9x^2-4}{x}\end{aligned}$$

(b)

$$\begin{aligned}(f-g)(x) &= (3x+2)-\frac{3x-2}{x}\\ &= \frac{x(3x+2)-(3x-2)}{x}\\ &= \frac{3x^2+2x-3x+2}{x}\\ &= \frac{3x^2-x+2}{x}\end{aligned}$$

EXAMPLE 2:

Given two functions $f(x)=\sqrt{x+2}$ and $g(x)=x^2$, find and simplify

(a) $(f\circ g)(x)]$

(b) $\left(\frac{f}{g}\right)(x)$

Solution: (a)

$$\begin{aligned}(f\circ g)(x) &= f[g(x)]\\ &= f(x^2)\\ &= \sqrt{x^2+2}\end{aligned}$$

(b)

$$\left(\frac{f}{g}\right)(x)=\frac{\sqrt{x+2}}{x^2}$$

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Example 3:

Let $f(x) = \sqrt{x+3}$ and $g(x) = 2x - 1$. Evaluate

(a) $(fg)\left(\frac{1}{2}\right)$

(b) $\left(\frac{f}{g}\right)(-2)$.

Solution: (a)

$$\begin{aligned}(fg)\left(\frac{1}{2}\right) &= f\left(\frac{1}{2}\right) \times g\left(\frac{1}{2}\right)\\ &= \sqrt{\left(\frac{1}{2}+3\right)} \times \left(2 \times \frac{1}{2} - 1\right)\\ &= \left(\sqrt{\frac{7}{2}}\right) \times 0\\ &= 0\end{aligned}$$

(b)

$$\begin{aligned}\left(\frac{f}{g}\right)(-2) &= \left(\frac{f(-2)}{g(-2)}\right)\\ &= \left(\frac{\sqrt{-2+3}}{-4-1}\right)\\ &= \left(\frac{1}{-5}\right)\\ &= -\left(\frac{1}{5}\right)\end{aligned}$$

Example 4:

Let $f(x) = 2x + 1$ and $g(x) = x - 2$. Find and simplify $(f \circ g)(x)$.

Solution:

$$\begin{aligned}(f \circ g)(x) &= f[g(x)] \\ &= f[x-2] \\ &= 2(x-2)+1 \\ &= 2x-4+1 \\ &= 2x-3\end{aligned}$$

Example 5:

Let $f(x) = x^2 + 4$ and $g(x) = \sqrt{x+1}$. Evaluate

(a) $(f \circ g)(3)$.

(b) $(g \circ f)(-2)$.

Solution: (a)

$$\begin{aligned}(f \circ g)(3) &= f[g(3)] \\ &= f[\sqrt{4}] \\ &= f[2] \\ &= 2^2 + 4 \\ &= 8\end{aligned}$$

(b)

$$\begin{aligned}(g \circ f)(-2) &= g[f(-2)] \\ &= g[8] \\ &= \sqrt{9} \\ &= \pm(3)\end{aligned}$$

Example 6:

Let $f(x) = \frac{1}{x-1}$ and $g(x) = \sqrt{x}$. Determine

(a) $(f \circ g)(x)$.

(b) $(g \circ f)(x)$.

Solution: (a)

$$\begin{aligned}(f \circ g)(x) &= f[g(x)] \\ &= f[\sqrt{x}] \\ &= \frac{1}{\sqrt{x}-1}\end{aligned}$$

(b)

$$\begin{aligned}(g \circ f)(x) &= g[f(x)] \\ &= g[\frac{1}{x-1}] \\ &= \sqrt{\frac{1}{x-1}} \\ &= \frac{1}{\sqrt{x-1}}\end{aligned}$$

From the above example, we note that $(f \circ g)(x) \neq (g \circ f)(x)$. Generally, composition of two functions is not commutative. That is $(f \circ g)(x) \neq (g \circ f)(x)$.

For the composite function $(f \circ g)(x)$ to be meaningful, $g(x)$ must be in the domain of $f(x)$. To find the domain of $(f \circ g)(x)$, we need to consider the domain of $f(x)$ and the domain of $g(x)$.

Example 7:

Let $f(x) = x^2 + 2$ and $g(x) = \sqrt{x}$. Determine the following composite functions and domains:

(a) $(f \circ g)(x)$.

(b) $(g \circ f)(x)$.

Solution:

(a)

$$\begin{aligned}(f \circ g)(x) &= f[g(x)] \\ &= f[\sqrt{x}] \\ &= x+2\end{aligned}$$

Domain of $f(x)$ is all real numbers, but the domain of $g(x)$ is are non-negative real numbers. This makes the domain of $f[g(x)]$ as all non-negative real numbers. So, domain of $(f \circ g)(x)$ is $[0, \infty)$.

(b)

$$\begin{aligned}(g \circ f)(x) &= g[f(x)] \\ &= g[x^2 + 2] \\ &= \sqrt{x^2 + 2}\end{aligned}$$

In this case, domain of the inside function $f(x) = x^2 + 2$ is all real numbers, whereas the domain of the outside function $g(x) = \sqrt{x}$ is all non-negative real numbers. We note that the domain of $(g \circ f)(x)$ is $(-\infty, \infty)$.

One important operation of function involves finding the **difference quotient** of a function.

For any given function $y = f(x)$, the expression $\frac{f(x+h)-f(x)}{h}$ is called the **difference quotient** of the function. This difference quotient is of great relevance in calculus.

We can split the process of finding the difference quotient into 3 steps:

1. Step 1: Evaluate $f(x+h)$
2. Step 2: Evaluate $f(x+h) - f(x)$
3. Step 3: Simplify $\frac{f(x+h)-f(x)}{h}$

Example 8:

Find the simplify the difference quotient for the function $f(x) = x^2 + x - 3$

Solution:

$$\begin{aligned}
f(x+h) &= (x+h)^2 + (x+h) - 3 \\
&= x^2 + 2hx + h^2 + x + h - 3 \\
f(x+h) - f(x) &= (x^2 + 2hx + h^2 + x + h - 3) - (x^2 + x - 3) \\
&= 2hx + h^2 + h \\
\frac{f(x+h) - f(x)}{h} &= \frac{h(2x + h + 1)}{h} \\
&= 2x + h + 1
\end{aligned}$$

Example 9:

Find and simplify the difference quotient for the function $f(x) = \sqrt{x}$

Solution: For this function, step 1 is not needed. So, we shall start with step 2:

$$\begin{aligned}
f(x+h) - f(x) &= \sqrt{x+h} - \sqrt{x} \\
\frac{f(x+h) - f(x)}{h} &= \frac{\sqrt{x+h} - \sqrt{x}}{h}
\end{aligned}$$

In order to simplify this expression so that the h in the denominator can be canceled, we multiply and divide the expression by the conjugate of the numerator:

$$\begin{aligned}
\frac{f(x+h) - f(x)}{h} &= \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \frac{x + h - x}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \frac{h}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \frac{1}{\sqrt{x+h} + \sqrt{x}}
\end{aligned}$$

Example 10:

Find and simplify the difference quotient for the function $g(x) = \frac{2}{x+1}$

Solution:

$$
\begin{aligned}
g(x+h) &= \frac{2}{x+h+1} \\
g(x+h)-g(x) &= \frac{2}{x+h+1} - \frac{2}{x+1} \\
&= \frac{2(x+1)-2(x+h+1)}{(x+h+1)(x+1)} \\
&= \frac{2x+2-2x-2h-2}{(x+h+1)(x+1)} \\
&= \frac{-2h}{(x+h+1)(x+1)} \\
\frac{g(x+h)-g(x)}{h} &= \frac{-2j}{h(x+h+1)(x+1)} \\
&= \frac{-2}{(x+h+1)(x+1)}
\end{aligned}
$$

Practice Problems

(1) If $f(x) = 3x + 1$ and $g(x) = x^2$, find a formula for $f + g$.

$(f+g)(x) = x^2 + 3x + 1$

(2) If $f(x) = |x - 2|$ and $g(x) = \sin x$,find a formula for $2g - 4f$.

$(2g - 4f)(x) = 2\sin x - 4|x - 2|$.

(3) If $h(x) = \sqrt{x-2}$ and $g(x) = \tan x$, find a formula for $hg - 2$

$(hg - 2g)(x) = \tan x\sqrt{x-2} - 2\tan x$

(4) If $f(x) = x^4$ and $p(x) = \sqrt{x+1}$, find a formula for $3fp$

$(3fp)(x) = 3x^4\sqrt{x+1}$.

(5) If $r(t) = \sqrt{t+1}$ and $s(t) = \sqrt{t-2}$, find a formula for $\left(\frac{g}{f}\right)$.

$\sqrt{\left(\frac{t-2}{t+1}\right)}$.

(6) If $f(x) = \sqrt{x+5}$ and $g(x) = 3x - 1$,evaluate $((f-g)(-1)$.

$(f-g)(-1) = 6$.

(7) If $g(x) = x^2 - x$ and $h(x) = 6 - x^2$, evaluate $2(gh)(2)$

8.

(8) If $f(x) = x^2 - 2$ and $g(x) = \sqrt{x+1}$, find $(f \circ g)(x)$ and henc

$(f \circ g)(x) = x - 1$ and $(f \circ g)(-2) = -3$.

(9) If $f(x) = \frac{1}{2x}$ and $g(x) = \frac{2}{3x}$, find $(g \circ f)(3)$

4.

(10) If $f(x) = \frac{x}{x+1}$ and $g(x) = 4 - x^2$, evaluate $(g \circ f)(1)$.

$\frac{15}{4}$

(11) If $f(x) = \frac{1}{x-1}$ and $g(x) = \frac{1}{x+2}$, find $(f \circ g)(x)$,and state its

$(f \circ g)(x) = -\frac{x+2}{x+1}$. Domain is $(-\infty, -1) \cup (-1, \infty)$

(12) If $f(x) = x^2 - 1$ and $g(x) = \sqrt{x+1}$, find $(g \circ f)(x)$ and sta

$(g \circ f)(x) = x$. Domain is all real numbers: $(-\infty, \infty)$.

(13) Given that $f(x) = x^2 - 4$ and $g(x) - 3x + 6$, find and simpli

$(fg)(x) = 3x^3 + 6x^2 - 12x - 24$. Domain is all real numbers: $(-\infty, \infty)$

(14) Given that $f(x) = \sqrt{4-x}$ and $g(x) = \sqrt{x+2}$, evaluate $(\frac{g}{f})$

$\sqrt{\frac{1}{2}}$

(15) Let $f(x) = x + 2$ and $g(x) = \frac{1}{x+2}$. Find and simplify
(a) $(f \circ g)(x)$
(b) $(g \circ f)(x)$. Also state the domains of these composite functi

(a) $(f \circ g)(x) = \frac{2x+5}{x+2}$. Domain is $(-\infty, -2) \cup (-2, \infty)$;
(b) $(g \circ f)(x) = \frac{1}{x+4}$. Domain is $(-\infty, -4) \cup (-4, \infty)$

(16) Find and simplify the difference quotient for the function $f(x$

-3

(17) Find and simplify the difference quotient for the function $f(x$

$2x + 2h - 3$

(18) Determine the difference quotient of the function $f(x) = x^3$

$3x^2 + 3xh + h^2$

(19) Determine the difference quotient of the function $f(x) = \frac{3}{2-x}$

$\frac{3}{(2-x-h)(2-x)}$

(20) Determine the difference quotient of the function $f(x) = \sqrt{x}$

$\frac{1}{\sqrt{x+h-3}+\sqrt{x-3}}$

(21) Find and simplify the difference quotient of the function $f(x)$

$\frac{-6}{(x+h-3)(x-3)}$

(22) Find and simpify the difference quotient of the function $f(x)$

$$\frac{x^2+hx-2x}{(x-1)(x+h-1)}$$