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# PRECALCULUS

## Chapter 1 - Functions and Graphs

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### 1.1: Definitions and representations

A relation between two sets $X$ and $Y$ is a correspondence between elements of set $X$ and the elements of set $Y$. This can also be defined as a set of points in a $xy$- plane, whose coordinates are $(x, y)$ such that $x \in X$ and $y \in Y$.

A function is a relationship or mapping of the elements of a set $X$ to the elements of a set $Y$ such that each element in the set $X$ is mapped to a unique element in the set $Y$, by a rule.

The elements of the set $X$ are called input values and the elements of the set $Y$ which correspond to any element of the set $X$ are called the output values.

The set $X$ is called the **Domain** and the the set $Y$ is called the **Codomain**. The set of all elments of $Y$ that correspond to any element of $X$ is called the **Range.** Obviously, Range is a subset of Codomain.

All relations are not functions. Only the relations which map each input value to a unique output value are functions.

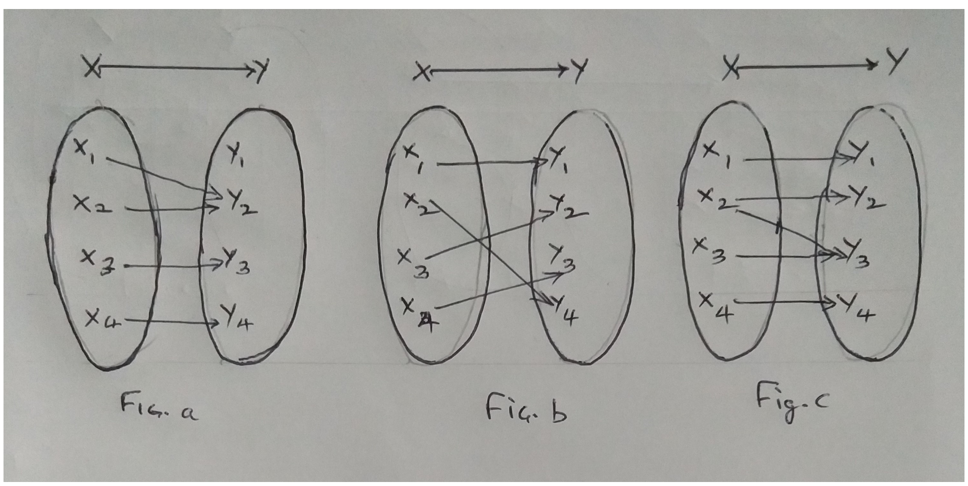
A function can also be viewed as a **Rule** that produces a unique element in the set of $Y$ for every element in the set of $X$, the elements of the set $Y$ is dependent on the elements of $X$. So, if we represent the elements of $Y$ as $y$, and the elements of $X$ as $x$, $x$ is called the **independent**

**variable** and $y$ is called the **dependent variable.**

A relationship between two sets $X$ and $Y$ can be represented by any of the following ways:

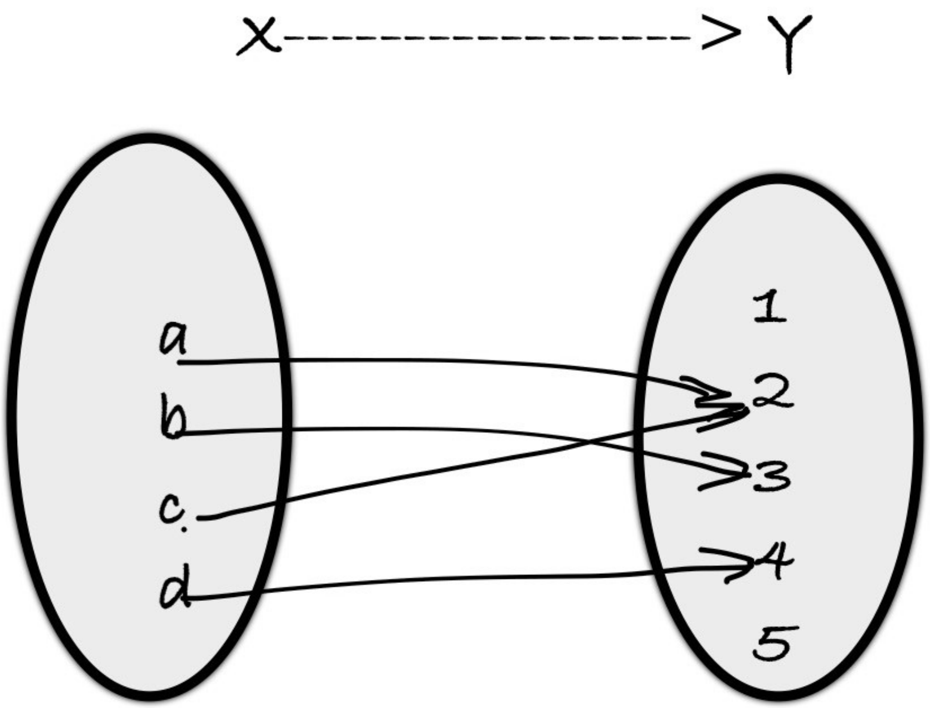
1. Diagramatic representation.
2. Algebraic representation.
3. Graphical representation.
4. Set rerpresentation, as a set of ordereed pairs.

**Diagramatic representation:** In the following diagram,
Figure(a) has more than one element of $X$ mapped to an element of $Y$. But this mapping conforms to the rule of a function. This type of function is called **Many-to-one function.**
Figure(b) has each element mapped to exactly one element in $Y$. This relationship also conforms the rule of a function. This type of function is called **One-to-one function.**
Figure(c) has at least one element of $X$ mapped to more than one element of $Y$. This does not conform to the rule of a function. So, these types of **one-to-many relationships are not functions .**




## EXAMPLE 1:

> Determine the type of relationship represented in the following diagram and state whether it is function or not:




*Solution*: As more than one element of $X$ is mapped to one element of $Y$, this is s many-to-one function.
Note that the domain of the function is the set $\{a, b, c, d\}$ and the range is $\{2, 3, 4\}$ while the codomain is $\{1, 2, 3, 4, 5\}$.

**Algrbraic representation:** The rule that associates the variable $x$ to the variable $y$ is written as $y = f(x)$,and read as "$y$ is a function of $x$.", or "$y$ is the value of $f$ at $x$. We can determine if a rule defines a function or not by checking if any value of $x$ gives more than one value for $y$ according to that rule.

## EXAMPLE 2:

Determine whether the formula $y = x^2 - 2x + 5$ defines a function.

*Solution*: Yes. Because, when we plug in any value for $x$ in the formula, we get a unique value of $y$.
As any value of $x$ plugged into the equation gives a unique real value of $y$, the domain of this function is the set of all real numbers $(-\infty, \infty)$ and the range is also the set of all real numbers $(-\infty, \infty)$

### EXAMPLE 3:

Determine if the formula $x^2 + y^2 - 2x = 6$ defines a function.

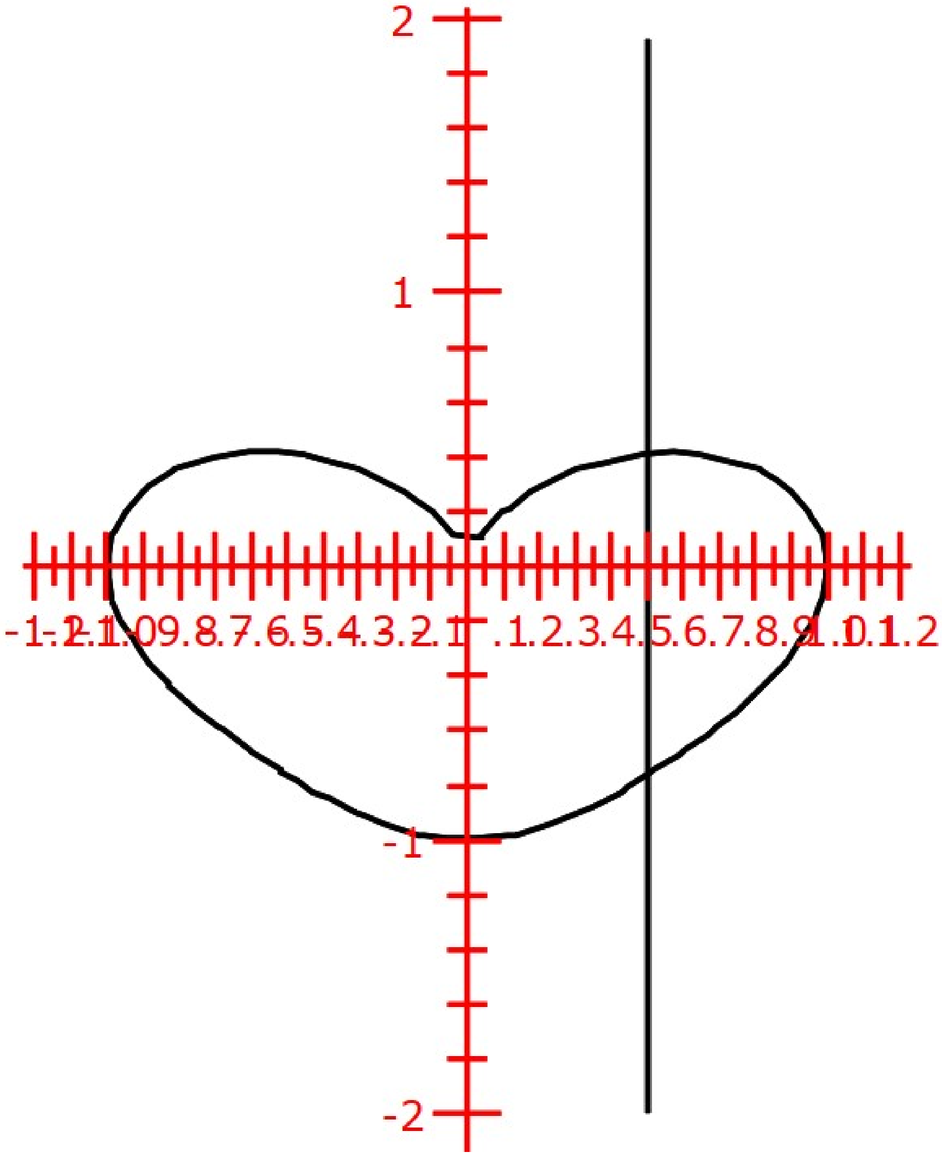
*Solution*: No. Solving this formula for $y$, we get

$$y^2 = 6 - x^2 + 2x$$
$$y = \pm\sqrt{6 - x^2 + 2x}$$

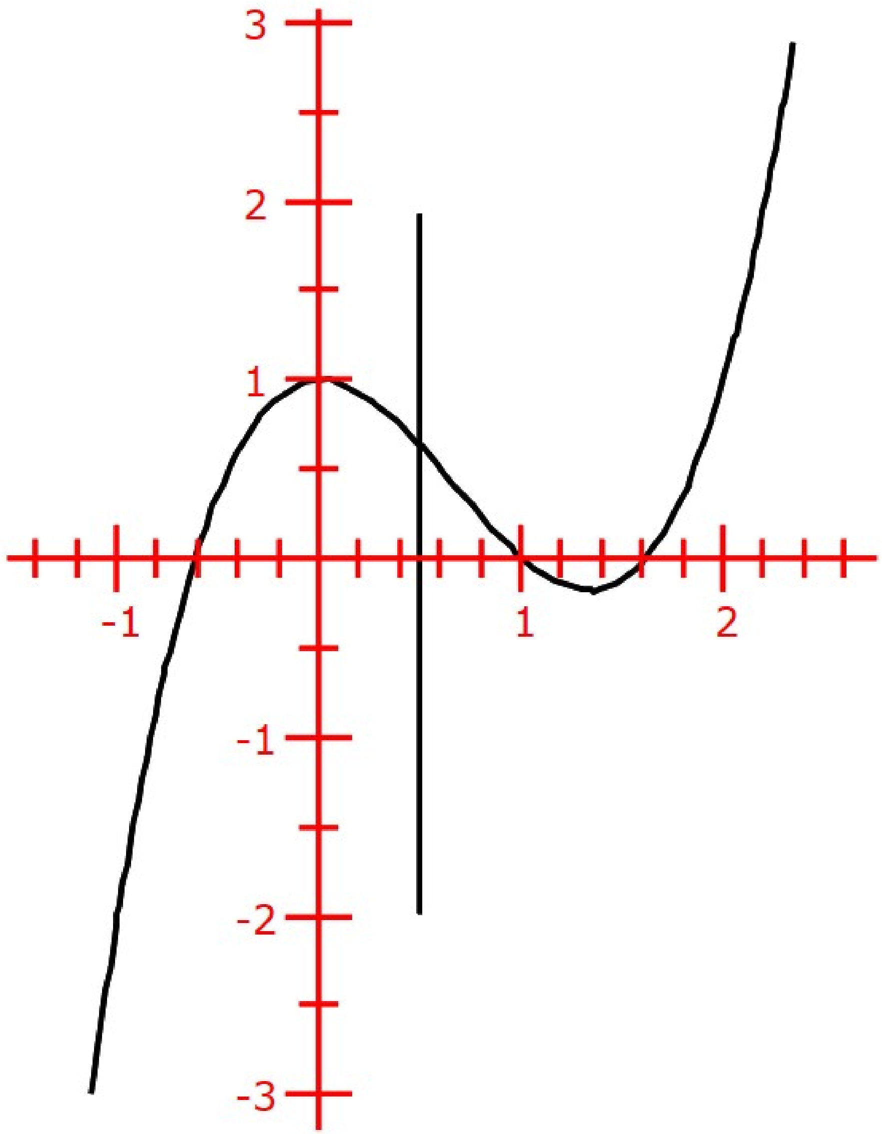
This means, for each value of $x$ we substitute in the formula, we get two values of $y$. So, this is not a function.

**Graphical representation:** Any relationship between $x$ and $y$ can be graphed by plotting ordered pairs $(x, y)$ that satisfy the relationship. A graph defines a function if and only if no vertical line intersects the graph in more than one point. If a vertical line intersects the graph in more than one point, this would mean that the graph gives more than one value for $y$ for that particular value of $x$ where that vertical line is placed. This test of determining a funtion using graph is called **vertical line test** .

Below are two graphs that confirm this rule.




In the above graph, the vertical line intersects the graph in two points. So, this is not the graph of a function.




The above graph is a function a vertical line crosses the graph in exactly one point. Note that this is true for any vertical line.

**Set representation:** A relation can be represented by a set of ordered pairs $(x, y)$ where the values of $x$ are in the domain and the values of $y$ are in the range. In this representation, we just need to check if any value of $x$ is paired with more than one value of $y$.

## EXAMPLE 4:

Check whether or not the set of ordered pairs $\{(1,2),(2,4),(5,1),(1,8)\}$ defines a funciton.

*Solution*: No. Note that $x = 1$ is paired with $y = 2$ and $y = 8$.

## Example 5:

Determine if the set $\{(4,1),(3,8),(1,2),(5,2)\}$ defines a function.

*Solution*: Yes. Note that each velue of $x$ is paired with a different value of $y$. Also note that the values $x = 1$ and $x = 5$ are paired with the value of $y = 2$. This is a many-to-one function.
Domain of this function is $1, 3, 4, 5$ and range is $1, 2, 8$.

The **domain** of a function is the set of all values of $x$ that makes the value of the function meaningful.
Unless otherwise stated or implied, domain of all polynomial functions is the set of real numbers $(-\infty, \infty)$.

The domain of a rational function is the set of all real numbers that do not make the demominator expression zero.

The domain of a radical function is the set of all realnumbers that make a non-negative radicand.
**Range** of a function is the set of all values of $y$ that correspond to the values of $x$.

## Example 6:

Find the domain and range of the function $y = \sqrt{x-2}$

*Solution*: To find the domain, set the radicand greater than or equal to zero, and solve.

$$x - 2 \geq 0$$
$$x \geq 2$$

So, the domain of this function is $\{x|x \in \mathcal{R} : x \geq 2\}$

In interval notation, domain is $[2, \infty)$.
The least value of $x$ is 2, which result in $y = 0$. All values of $y$ are positive. So, the range of this function is $\{y|y \in \mathcal{R} : y \geq 0\}$. In interval notation, range is $[0, \infty)$.

## Example 7:

Find the domain and range of the function $y = \frac{x+2}{x-3}$.

*Solution*: To find the domain, set the denominator to zero, and solve.Then, exclude thIs value from the set of real numbers.

$$\begin{aligned} x - 3 &= 0 \\ x &= 3 \end{aligned}$$

So the domain is $\{x|x \in \mathcal{R} : x \neq 0\}$ and in interval notation, $(-\infty, 0) \cup (0, \infty)$
Range is the set all real numbers $(-\infty, \infty)$

## Example 8:

Find the domain and range of the function $f(x) = \frac{2x-3}{\sqrt{x+1}}$

*Solution*: Domain: While numerator can take all values, denominator cannot be zero, and cannot be negative because of the square-root. So, the domain would be all real numbers that satisfies the inequality $x + 1 > 0$. Solving this would result in all values of $x > -1$. So, domain of this function is $\{x|x \in \mathcal{R} : x > -1\}$ in set notation and $(-1, \infty)$ in interval notation.
Range: As the denominator is always positive and can be either less than or more than $1$, the range is all real numbers $\{x|x \in \mathcal{R}\}$ in set notation and $(-\infty, \infty)$ in interval notation.

As the algebraic notation of a function $y = f(x)$ suggests, the variable $x$ is the input value of the function. The value of the function is the output value $y$, which is the result of substituting the value of $x$ in the expression f(x).

### Example 9:

Find the value of the function $y = x^2 - 3$ for $x = -2$.

*Solution*: Substituting the $x = -2$ in the function, we get

$$\begin{aligned} y &= (-2)^2 - 3 \\ &= 4 - 3 \\ &= 1 \end{aligned}$$

### Example 10:

A function is defined by $f = \sqrt{1 - 2x}$. Find the point where the graph of this function intersects $x$-axis..

*Solution*: Any point on the $x$-axis has its $y$ coordinate $0$. So, the x-intercepts are obtained by solving $f(x) = 0$.

$$\begin{aligned} \sqrt{1 - 2x} &= 0 \\ 1 - 2x &= 0 \\ 1 &= 2x \\ \frac{1}{2} &= x \end{aligned}$$

### Example 11:

A function is defined by the set of ordered pairs $f = \{(-2, 3), (-1, 0), (0, 1), (1, 0), (2, 1)\}$. Find f(-1).

*Solution*: Note that the ordered pair that has $x$- coordinate $-1$ has $y$ -coordinate $0$.So, $f(-1) = 0$.

# Practice Problems

(1) Determine whether the following set diagram defines a function and if so, if it is one-to-one or many-to-one function: 1

(2) Determine whether the following set diagram defines a function and if so, if it is one-to-one or many-to-one function:

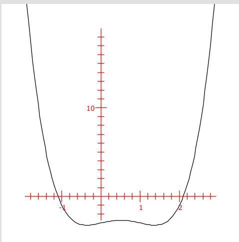
(3) Determins whether the relationship $y + 2x - 3 = 0$ is a function or not. If it is a function, is it one-to-one or many-to-one?

(4) Determine whether the relationship $^2 + 2x - y + 3 = 0$ is a function or not.If it is a function, is it one-to-one or many-to-one?

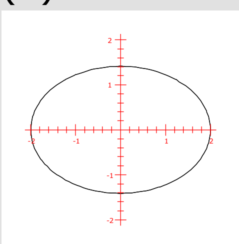
(5) Determine whether the relationship $x - 2y^2 + 3 = 0$ is a function or not.If it is a function, is it one-to-one or many-to-one?

(6) Use the vertical line test to determine whether the given graph is

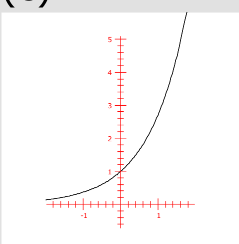
(a)



(b)



(c)



(7) Determine whether each of the following collection of ordered pairs represent a function or not. If it is a function, state if it is one-to-one or many-to-one function.

$$f = \{(1,2),(2,3),(6,3),(10,4),(14,20)\}$$
$$g = \{(3,1),(1,3),(2,5),(3,6)\}$$
$$h = \{(10,2),(8,8),(7,10),(6,12),(2,4)\}$$

(8) Find the domain and range of the function
$f = (1,2),(2,4),(4,8),(5,9),(8,13)$

(9) Find the domain of the function $f(x) = \frac{4}{2x+3}$

(10) Find the domain of the function $f(x) = \sqrt{x^2 - 4}$

(11) Fin the domain of the function $g(x) = \frac{x-3}{(x+2)(x-1)}$

(12) Find the domain and range of the function $x^2 - 5$.

(13) Find the domain and range of the function $g(x) = \frac{2-x}{x^2-9}$

(14) Find the domain and range of the function $g(x) = \frac{x^2}{\sqrt{1-x^2}}$

(15) Find the value of the function $f(x) = 2x^3 - 3x + 2$ when $x = 2$.

(16) Given that $h(x) = \frac{1-x}{x^3}$, find $h(-1)$

(17) A function $f$ is defined the set of ordered pairs $\{(1,3),(2,4,(3,3),(4,0),(5,-5)$.Find $f(2)$

(18) Given that $f(x) = x^2 - 1$ and $g(x) = 1 - 2x$, find $f[g(-2)]$

(19) Does the collection of ordered pairs $\{(-2,1),(1,3),(2,4),(3,1)\}$ define a function? If yes, state its domain and range.

(20) Determine which of the following equations represent $y$ as a function of $x$:
(a) $x^3 + y^2 = 1$;
(b) $x^2 + y^3 = 2$;
(c) $x^y + 3y = 1$

(21) A function is defined by $y = -x^2 + 3x + 4$. Simplify the following:
(a) $f(2x)$;
(b) $2f(x)$;
(c) $f(x+2)$;
(d) $f(x) + f(2)$.

(22) Find the domain of the function $f(x) = \sqrt{5-3x}$

(23) Find the domain of the function $\frac{\sqrt{2x+1}}{x^2-1}$

(24) Given a function $f(x) = \sqrt{2+1}$, evaluate $f(a+h)$

(25) For a function $g(x) = x^2 - x - 12$, find the values of $x$ for which the graph of the function intersects the $x$-axis.