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PRECALCULUS

Chapter P - Preliminary Concepts

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P7: QUADRATIC EQUATIONS

A quadratic equation in one variable is of the form $ax^2 + bx + c = 0$ where a, b,and c are real numbers and $a \neq 0$.
To solve a quadratic equation means to find all values of x for which the equation is true. As this is a second degree equation, we have at most two values that would satisfy the equation. So, we call the pair of values as solution set.
To solve a quadratic equation $ax^2 + bx + c = 0$ also means to find the x -intercepts of the graph of $y = ax^2 + bx + c$. The values of the solutions are also called the roots or zeros of the equation.

Generally, a quadratic equation can be solved algebraically in any one of the following methods:

1. By factoring, using zero factor property. (If $ab = 0$, then, $a = 0$ or $b = 0$)
2. By taking square root on both sides
3. By completing the square
4. By using quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

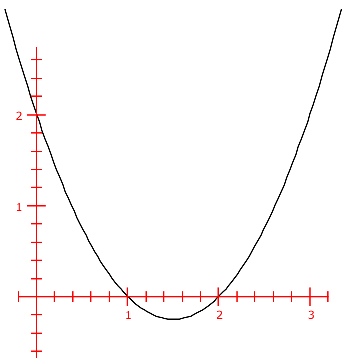
EXAMPLE 1:

Solve the equation $x^2 - 3x + 2 = 0$

Solution: Algebraically, we can factor the left side and set each factor to zero. Then solve each resulting linear equation to get the solution set.

$$\begin{aligned} x^2 - 3x + 2 &= 0 \\ (x-1)(x-2) &= 0 \\ x - 1 = 0 \ or \ x - 2 &= 0 \\ x = 1 \ or \ x &= 2 \end{aligned}$$

So, solution set is {1,2}
Graphically, we can sketch the graph of the equation $y = x^2 - 3x + 2$ and determine the x intercepts of this graph.



EXAMPLE 2:

Solve $(2x+1)^2 = 9$ by square roots method.

Solution: We solve it by taking square root on both sides:

$$\begin{aligned} (2x+1)^2 &= 9 \\ 2x + 1 &= \pm 3 \\ 2x + 1 = 3 \quad or \quad 2x + 1 &= -3 \\ 2x = 2 \quad or \quad 2x &= -4 \\ x = 1 \quad or \quad x &= -2 \end{aligned}$$

EXAMPLE 3:

Solve $3(2x-1)^2 - 4 = 0$ by square roots method.

Solution: We do this by rewriting the equation in the form $(X)^2 = A$

where (X) is an expression in x and A is a constant. Then, we solve it by taking square root on both sides, like the previous problem.

$$\begin{aligned}
3(2x-1)^2 - 4 &= 0 \\
3(2x-1)^2 &= 4 \\
(2x-1)^2 &= \frac{4}{3} \\
2x-1 &= \pm\sqrt{\frac{4}{3}} \\
2x-1 &= \frac{2}{\sqrt{3}} \quad or \quad 2x-1 = -\frac{2}{\sqrt{3}} \\
2x-1 &= \frac{2\sqrt{3}}{3} \quad or \quad 2x-1 = -\frac{2\sqrt{3}}{3} \\
2x &= 1+\frac{2\sqrt{3}}{3} \quad or \quad 2x = 1-\frac{2\sqrt{3}}{3} \\
x &= \frac{1}{2}+\frac{\sqrt{3}}{3} \quad or \quad x = \frac{1}{2}-\frac{\sqrt{3}}{3}
\end{aligned}$$

EXAMPLE 4:

Solve $x^2+6x-7=0$ by completing the square method.

Solution:

$$x^2+6x-7=0$$

Take the constant term to the right side:

$$x^2+6x=7$$

Complete the square:

$$\begin{aligned}
x^2+6x+3^2 &= 7+3^2 \\
(x+3)^2 &= 16
\end{aligned}$$

Now we can solve this by taking square root on both sides:

$$\begin{aligned} x+3 &= \pm 4 \\ x+3=4 \quad &or \quad x+3 = -4 \\ x=1 \quad &or \quad x = -7 \end{aligned}$$

EXAMPLE 5:

Solve $2x^2 - 7x + 9 = (x-3)(x+1) + 3x$ by completing the square method.

Solution: Rewrite the equation by simplifying the right side:

$$\begin{aligned} 2x^2 - 7x + 9 &= x^2 - 2x - 3 + 3x \\ 2x^2 - 7x + 9 &= x^2 + x - 3 \end{aligned}$$

Take all terms containing the variable to the left side, all constant terms to the right side, and simplify:

$$\begin{aligned} 2x^2 - x^2 - 7x - x &= -3 - 9 \\ x^2 - 8x &= -12 \end{aligned}$$

Complete the square:

$$\begin{aligned} x^2 - 8x + 4^2 &= -12 + 4^2 \\ (x-4)^2 &= 4 \end{aligned}$$

Take square root on both sides and solve for x:

$$\begin{aligned} x-4 &= \pm 2 \\ x-4=2 \quad &or \quad x-4 = -2 \\ x=6 \quad &or \quad x = 2 \end{aligned}$$

EXAMPLE 6:

Solve $2x^2 + 4x + 52 = 0$ by completing the square method.

Solution:
Step 1: Take the constant term to the right side:

$$2x^2 + 4x = -52$$

Step 2: Divide both sides by 2 in order to make the coefficient of $x^2 = 1$:

$$x^2 + 2x = -26$$

Step 3: Complete the square:

$$\begin{aligned} x^2 + 2x + 1 &= -25 \\ (x+1)^2 &= -25 \end{aligned}$$

Step 4: Take square root on both sides and solve for x:

$$\begin{aligned} x + 1 &= \pm 5i \\ x &= -1 + 5i \ or \quad x = -1 - 5i \end{aligned}$$

EXAMPLE 7:

Solve $2x^2 - 3x + 1 = 0$ by using the quadratic formula.

Solution: Here, $a = 2$, $b = -3$, and $c = 1$. Substitute these values in the quadratic formula and simplify:

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(1)}}{2(2)}$$
$$x = \frac{3 \pm \sqrt{9-8}}{4}$$
$$x = \frac{3 \pm 1}{4}$$
$$x = \frac{4}{4} \qquad or \quad x = \frac{2}{4}$$
$$x = 1 \qquad or \quad x = \frac{1}{2}$$

EXAMPLE 8:

Solve $x^2 + 4x - 6 = 0$ by using the quadratic formula.

Solution: Here, $a = 1$, $b = 4$, and $c = -6$. Substitute these values in the quadratic formula and simplify:

$$x = \frac{-(4) \pm \sqrt{(4)^2 - 4(1)(-6)}}{2(1)}$$
$$x = \frac{-4 \pm \sqrt{16+24}}{2}$$
$$x = \frac{-4 \pm 2\sqrt{10}}{2}$$
$$x = -2 + \sqrt{10} \qquad or\ x = -2 - \sqrt{10}$$

The quadratic formula for solving the equation $ax^2 + bx + c = 0$ is $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. In this formula, the radicand $\sqrt{b^2 - 4ac}$ is called the discriminant because it helps us to discriminate the nature of roots of the equations.

1. If $b^2 - 4ac > 0$, there are two distinct real roots to the equation. This is because $\sqrt{b^2 - 4ac}$ would result in two distinct real values.
2. If $b^2 - 4ac = 0$ there is one repeated real root to the equation. This

is because, in this case the solution becomes $x = \frac{-b}{2a}$.

3. If $b^2 - 4ac < 0$, then there are two distinct complex roots to the equation. These two roots are complex conjugates of each other. This is because the negative radicand results in the imaginary part of the roots.

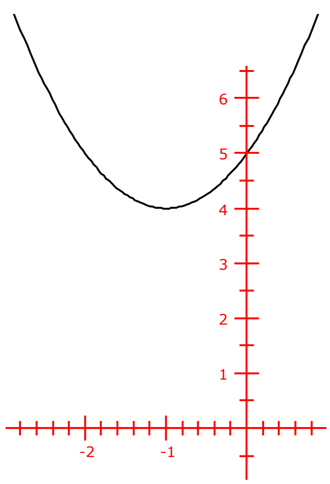
Example 9 :

Determine the nature of zeros of the equation $x^2 + 2x + 5$ without actually solving it.

Solution: Here, $a = 1$, $b = 2$, and $c = 5$. So,the discriminant is:

$$\begin{aligned} 2^2 - 4(1)(5) &= 4 - 20 \\ &= -16 \quad < 0 \end{aligned}$$

So,this equation has a complex conjugate pair of roots. No real roots implies that the graph does not intersect x-axis.



Example 10 :

Solve: $3x^2 + x + 2 = 0$

Solution: Substitute $a = 3$, $b = 1$, and $c = 2$ in the quadratic formula and simplify:

$$x = \frac{-1 \pm \sqrt{1^2 - 4(3)(2)}}{2(3)}$$

$$x = \frac{-1 \pm \sqrt{-23}}{6}$$

$$x = \frac{-1 + i\sqrt{23}}{6} \qquad or \quad x = \frac{-1 - i\sqrt{23}}{6}$$

$$x = -\frac{1}{6} + \frac{i\sqrt{23}}{6} \qquad or \; -\frac{1}{6} - \frac{i\sqrt{23}}{6}$$

Practice Problems

Solve problems #1 - #6 by factoring:

(1) $x^2 - 8x + 15 = 0$

(2) $2x^2 + 5x - 3 = 0$

(3) $x(3x + 11) = 20$

(4) $2p^2 - 3p - 2 = 0$

(5) $3 - 2x - x^2 = 0$

(6) $15t^2 - 16t + 4 = 0$

Solve problems #7 - #12 by taking square root on both sides:

(7) $(3x-4)^2 = 16$

(8) $8(t+1)^2 = 18$

(9) $(5x-8)^2 = 49$

(10) $(3x+4)^2 = 6$

(11) $(x+5)^2 + 4 = 0$

(12) $(z-3)^2 + 5 = 0$

Solve problems #13 - #16 by completing the square method:

(13) $3x^2 - 6x - 10 = 3x - x(x+1)$

(14) $x^2 - 6x - 3 = 0$

(15) $2x^2 + 2x + 5 = 0$

(16) $x^2 - 4x + 2 = 0$

Solve problems #17-22 using the quadratic formula

(17) $3x^2 = 6x + 5$

(18) $2x^2 - 7x + 4 = 0$

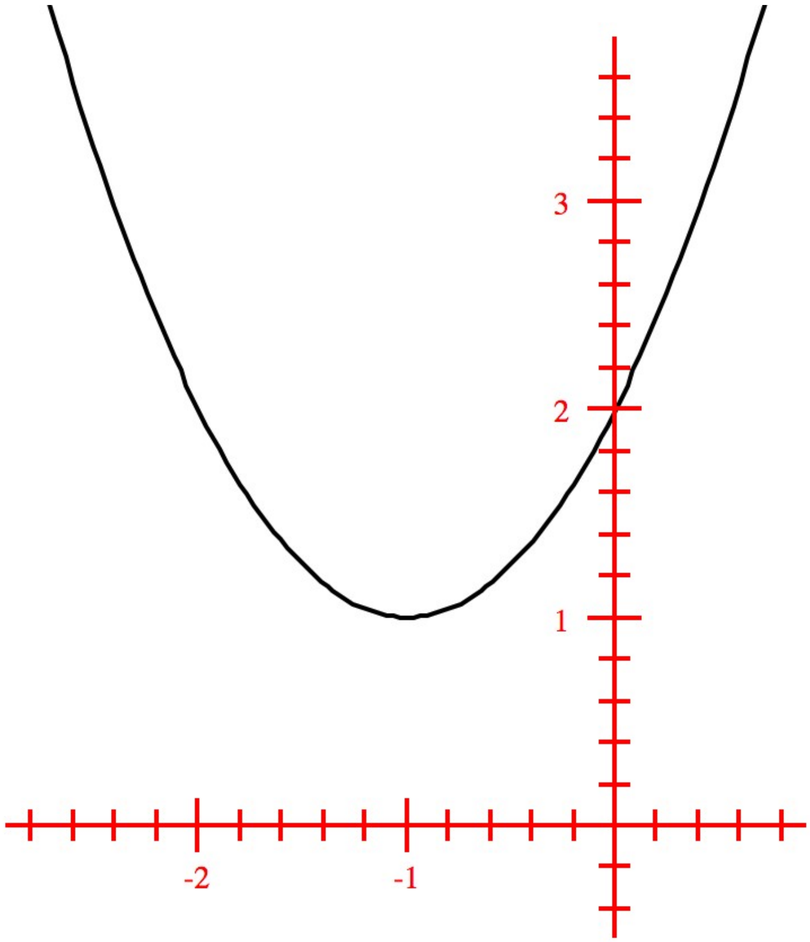
(19)$x^2 - 2x + 6 = 2x^2 - 6x - 26$

(20) $x^2 - 2x + 2 = 0$

(21) $3x^2 + x - 1 = 0$

(22) $5x^2 = 6x - 3$

(23) Solve the equation $x^2 + 2x = -2$ graphically.



(24) Determine and number and nature of roots of the quadratic equation $3x^2 + 6x - 6 = x^2 + 3x - x(x + 1) + 3$ using its discriminant.

(25) Determine and number and nature of roots of the quadratic equation $x^2 + 4 = 4x$ using its discriminant

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