**_Dr. Lalitha Subramanian_**
**_About the Author_**

Aksharam

# PRECALCULUS

## Chapter P - Preliminary Concepts

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## P6: COMPLEX NUMBERSs

The number $i = \sqrt{-1}$ is defined as the *imaginary unit*.
For any negative number $a$, $\sqrt{a} = \sqrt{|a|} \times i$.

### EXAMPLE 1:

Simplify $\sqrt{-12}$

*Solution:*

$$\begin{aligned} \sqrt{-12} &= \sqrt{4 \times -3} \\ &= 2 \times \sqrt{-3} \\ &= 2i\sqrt{3} \end{aligned}$$

A complex number is any number that can be written in the form of $a + bi$ where $a$ and $b$ are real numbers. This expression consists of two parts: The first part $a$ is the real part and the second part $b$ is the imaginary part. The expression $a + bi$ is called the standard form of writing a complex number.

Following are the rules of addition and subtraction with complex numbers:
If $a + bi$ and $c + di$ are two complex numbers, then,

1. **Sum** of two complex numbers:

$$(a+bi)+(c+di)=(a+c)+(b+d)i$$

2. **Difference** between two complex numbers:

$$(a+bi)-(c+di)=(a-c)+(b-d)i$$

### Example 2:

Add: $(5-3i)+(1+4i)$

*Solution:*

$$\begin{aligned}(5-3i)+(1+4i)&=(5+1)+(-3+4)i\\&=6+i\end{aligned}$$

### Example 3:

Subtract: $(-2+3i)-(1-5i)$

*Solution:*

$$\begin{aligned}(-2+3i)-(1-5i)&=(-2-1)+(3+5)i\\&=-3+8i\end{aligned}$$

When we multiply a complex number by a constant, we multiply both real and imaginary parts by that constant.

### Example 4:

Simplify: $2(5-4i)-3(1+i)$

*Solution:*

$$\begin{aligned}2(5-4i)-3(1+i)&=(10-8i)-(3+3i)\\&=10-8i-3-3i\\&=10-3-8i-3i\\&=7-11i\end{aligned}$$

Based on the definition of the imaginary unit $i$, we have

$i^2 = (\sqrt{-1}^2) = -1$. Thus, we see that square of an imaginary number becomes a real number.

When we multiply two complex numbers, we use the same method we follow while multiplying two binomials. That is, if $a + bi$ and $c + di$ are two complex numbers,

$$\begin{aligned}(a+bi)(c+di) &= a(c+di) + bi(c+di)\\ &= ac + adi + bci + bdi^2\\ &= ac + (bc+ad)i - bd\\ &= (ac-bd) + (bc+ad)i\end{aligned}$$

## Example 5:

Multiply: $(3+2i)(4-i)$

*Solution:*

$$\begin{aligned}(3+2i)(4-i) &= 3(4-i) + 2i(4-i)\\ &= 12 - 3i + 8i - 2i^2\\ &= 12 + 5i + 2\\ &= 14 + 5i\end{aligned}$$

The product of two complex numbers $a + bi$ and $a - bi$ is $(a+bi)(a-bi) = a^2 - (bi)^2 = a^2 + b^2$, which is a real number. For this reason, these two numbers $z = a + bi$ and $\overline{z} = a - bi$ are called complex conjugates.

When we divide two complex numbers, we basically eliminate the complex number in the denominator by multiplying and dividing the expression by the complex conjugate of the denominator, and then simplify the resulting expression.

## Example 6:

Simplify: $\frac{2-3i}{4i}$

*Solution:*

$$
\begin{aligned}
\frac{2-3i}{4i} &= \frac{(2-3i)(i)}{(4i)(i)} \\
&= \frac{2i-3i^2}{4i^2} \qquad = \frac{3+2i}{-4} \\
&= -\frac{3+2i}{4} \\
&= -\frac{3}{4} - \frac{1}{2}i
\end{aligned}
$$

## Example 7:

Write the complex number $\frac{3+i}{2-3i}$ in standard form

*Solution:*

$$
\begin{aligned}
\frac{3+i}{2-3i} &= \frac{(3+i)(2+3i)}{(2-3i)(2+3i)} \\
&= \frac{6+9i+2i+3i^2}{4-9i^2} \\
&= \frac{3+11i}{13} \\
&= \frac{3}{13} + \frac{11}{13}i
\end{aligned}
$$

## Example 8:

Simplify: $\frac{2}{4-3i}$

*Solution:*

$$\begin{aligned}\frac{2}{4-3i} &= \frac{2(4+3i)}{(4-3i)(4+3i)} \\ &= \frac{8+6i}{16+9} \\ &= \frac{8+6i}{25} \\ &= \frac{8}{25}+\frac{6}{25}i\end{aligned}$$

Properties of operations of complex numbers:
If $z$, $p$, and $w$ are three complex numbers, then

1. **Commutative property:** Addition and multiplication are commutative whereas subtraction and division are not commutative. That is, $z+w=w+z$, but $z-w\neq w-z$. In fact, $z-w=-(w-z)$. Similarly, $zw=wz$ but $\frac{z}{w}\neq\frac{w}{z}$. In fact, $\frac{z}{w}=\frac{1}{\frac{w}{z}}$
2. **Associative property:** $z+(w+p)=(z+w)+p$. Note that this may not work if there is a negative sign between any two numbers. In particular, $z+(w-p)=(z+w)-p$ but $z-(w+p)\neq(z-w)+p$. In fact, $z-(w+p)=(z-w)-p$. Similarly, $z(wp)=(zw)p$. But $\frac{z}{wp}\neq\frac{zw}{p}$
3. **Additive identity** of complex number system is $0=0+0i$. **Additive inverse** of $a+bi$ is $-a-bi$ because

$$\begin{aligned}(a+bi)+(-a-bi) &= (a-a)+(b-b)i \\ &= 0+0i \\ &= 0\end{aligned}$$

4. **Multiplicative identity** of complex number system is $1=1+0i$. **Multiplicative inverse** of $a+ib$ is $\frac{1}{a+ib}$ because

$$\begin{aligned}(a+ib)\times\frac{1}{a+ib} &= \frac{a+ib}{a+ib} \\ &= 1\end{aligned}$$

### EXAMPLE 9:

Simplify: $(2-i)+3(1-i)-2(5+7i)$

*Solution:*

$$\begin{aligned}(2-i)+3(1-i)-2(5+7i) &= 2-i+3-3i-10-14i \\ &= 2+3-10-i-3i-14i \\ &= -5-18i\end{aligned}$$

The imaginary unit $i$ is an interesting unit. We saw earlier that $i^2=-1$. Now, let us determine the higher powers of $i$.

$$\begin{aligned}
i^3 &= i^2 \times i = -i \\
i^4 &= i^2 \times i^2 && = (-1)(-1) = 1 \\
i^5 &= i^4 \times i && = (1) \times i && = i \\
i^6 &= i^4 \times i^2 && = (1)(-1) && = -1 \\
i^7 &= i^4 \times i^3 && = (1)(-i) && = -1 \\
i^8 &= i^4 \times i^4 && = (1)(1) && = 1
\end{aligned}$$

From the above, we observe that the imaginary unit $i$ is a cyclic number or order 4. That is, the same values of $i, -1, -i, 1$ are repeated for every four powers of $i$.

### EXAMPLE 10:

Simplify: $i^7-2i^4+4i^2$.

*Solution:*

$$\begin{aligned}i^7-2i^4+4i^2 &= i-2-4 \\ &= -6-i\end{aligned}$$

If two complex numbers are equal, then their real and imaginary parts must be correspondingly equal. That is, if $a+bi=c+di$, then $a=c$ and $b=d$.

### Example 11:

Find two real numbers $x$ and $y$ such that
$(5-3i)+(3-i)=x-(3+yi)$

*Solution:*

$$\begin{aligned}((5-3i)+(3-i) &= x-(3+yi)\\ 5+3-3i-i &= x-3-yi\\ 8-4i &= (x-3)-yi\end{aligned}$$

Setting the real and imaginary parts of both sides correspondingly equal,

$$\begin{aligned}x-3=8 \quad -y &= -4\\ x=11 \quad y &= 4\end{aligned}$$

## Practice Problems

For problems #1 - #16, simplify each expression and write the answer in standard form.

(1) $(2+3i)+(-3+i)$

(2) $(4-i)+(3+5i)$

(3) $(2-6i)+(3-2i)$

(4) $(2+3i)-(8i-2)$

(5) $(i^2 - 2) - (5 + i^3)$

(6) $(3 - \sqrt{-16}) + (\sqrt{-81} - 2)$

(7) $(2 - i)(4 + 3i)$

(8) $(2 - 4i)(3 - 2i)$

(9) $(\sqrt{-4} + 2i)(6 + 5i)$

(10) $2\sqrt{-9} + 5\sqrt{-25} - 8i^2 + 4i^6$

(11) $\frac{1}{2-3i}$

(12) $(1 - 2i)^2$

(13) $\frac{1-i}{3i}$

(14) $(2 + i)^3$

(15) $\frac{(2-i)(1+2i)}{1-2i}$

(16) $\frac{3+2i}{3-2i}$

(17) Find the real numbers $x$ and $y$ such that $x + 6i = (3 - i) + (5$

(18) Find the value of real numbers $a$ and $b$ such that $3 + 2bi = a -$

(19) Let $z = 2 + i$ and $w = 4 - 3i$. Then write the expression $2z -$

(20) Let $z = 1 - 2i$ and $w = 3 + i$. Then write the expression $z\bar{z} +$

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