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About the Author

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PRECALCULUS

Chapter P - Preliminary Concepts

PDF Version

P4: LINES IN A PLANE

Given two points P(x_1, y_1) and Q(x_2, y_2), distance between P and Q is given by the formula

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

EXAMPLE 1:

Find the distance between the points (1, 5) and (6, - 2)

Solution:

$$\begin{aligned} d &= \sqrt{(1-6)^2 + (5-(-2))^2} \\ &= \sqrt{(-5)^2 + (7)^2} \\ &= \sqrt{25+49} \\ &= \sqrt{79} \\ &= 8.89 \end{aligned}$$

Coordinates of the mid point of the line segment PQ is given by the formula

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

EXAMPLE 2:

Find the coordinates of the midpoint of the line segment joining P(-5, 2) and Q(3, 7).

Solution:

$$\begin{aligned}(x, y) &= \left(\frac{-5+3}{2}, \frac{2+7}{2}\right) \\ &= (-1, 4.5)\end{aligned}$$

The SLOPE of a line through the points P(x_1, y_1) and Q(x_2, y_2) is given by the formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

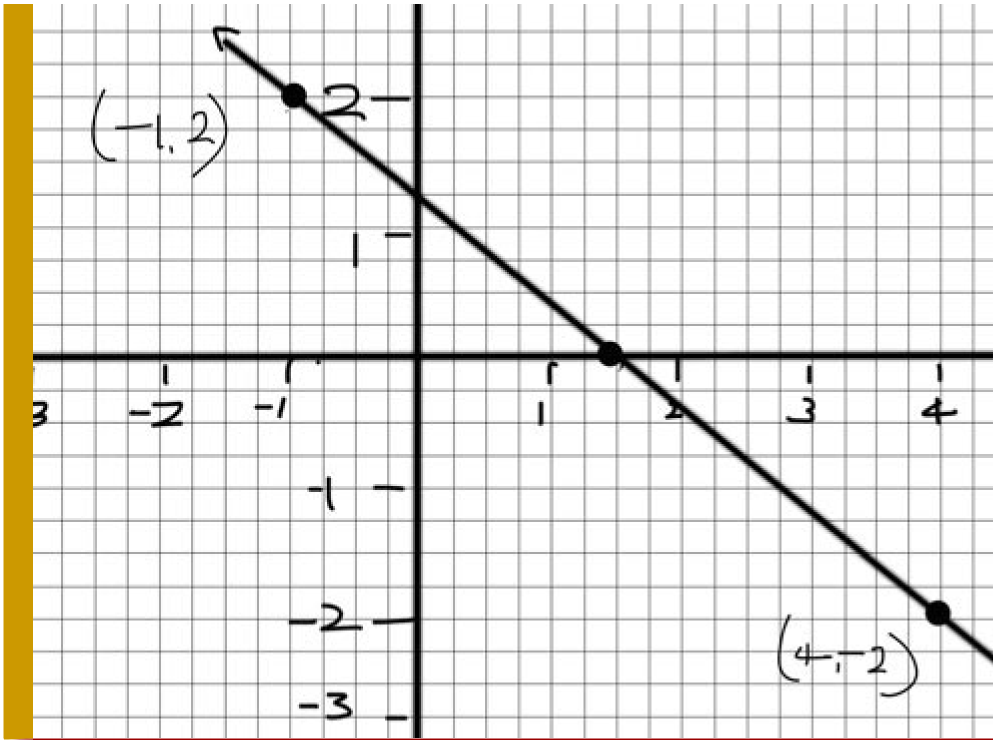
.

EXAMPLE 3:

Find the slope of the line through the points P(-1, 2) and Q(4, -2) and sketch the line.

Solution:

$$\begin{aligned}m &= \frac{(-2) - 2}{4 - (-1)} \\ &= -\frac{4}{5}\end{aligned}$$



An equation of a graph is an equation in two variables x and y, where x and y represent the x and y coordinates of any point on that graph. There is a one-one relationship between the coordinates of any point on the graph and the ordered pairs (x, y) that satisfy the equation. That is, coordinates of any point on the line should satisfy the equation of the line and any order pair of numbers (x, y) that satisfies the equation should represent a point on the graph.

A linear equation in x and y always represents the graph of a line.

DIFFERENT FORMS OF EQUATIONS OF A LINE

1. **Point-Slope Form** of an equation of a line with slope m passing through the point (x_1, y_1) is

$$y - y_1 = m(x - x_1)$$

2. **Slope-Intercept Form** of an equation of a line with slope m and y-intercept $(0, b)$ is

$$y = mx + b$$

3. **Two-Intercepts Form** of an equation of a line with x-intercept

$(a, 0)$ and y-intercept $(0, b)$ is

$$\frac{x}{a} + \frac{y}{b} = 1$$

4. **General Form** of an equation of a line

$$Ax + By + C = 0$$

   where A and B are not both zero
5. Equation of a **horizontal line** b units away from the x-axis is

$$y = b$$

   The line is b units above the x-axis if b is positive and b units below the x- axis if b is negative.
6. Equation of a **vertical line** a units away from y- axis is

$$x = a$$

   The line is a units to the right of y-axis if a is positive and a units to the left of y-axis if a is negative

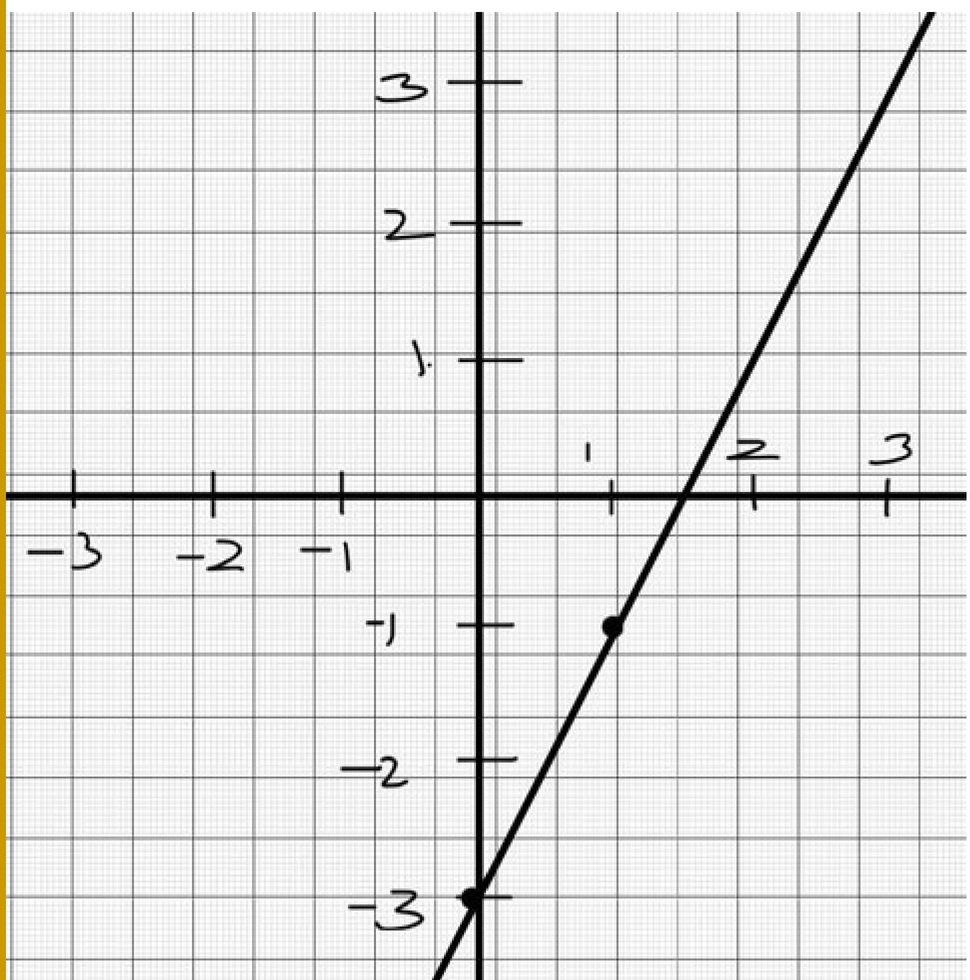
Graphing a line is easier if we remember the basic property of the line: **"Two distinct points determine a unique line."**

So, if we can find two distinct points satisfying a given equation, we can join them and extend on both sides to get the graph of the line. Alternately if we know the slope and y- intercept of a line, we can use the fact that $slope = \frac{rise}{run}$ and graph the line.

Example 4:

Graph the line $y = 2x - 3$

Solution: This equation is given in slope-intercept form. Slope of the line is 2 and y-intercept is -3. We can fix the first point on the y-axis as $(0, -3)$ to show the y- intercept. To get the second point, we use the definition of slope as $\frac{rise}{run}$. Here, slope is $2 = \frac{2}{1}$. That is, from the point $(0, -3)$, go up vertically by 2 units (rise) and horizontally 1 by unit (run), to get the second point $(1, -1)$.



Alternately, we can graph the line by getting the two intercepts. The y- intercept is obtained by setting $x = 0$ in the equation and solving for y. We can get the x- intercept by substituting y = 0 in the equation and solving for x. Remember that all points on the x- axis have y-coordinate zero. Similarly all points on the y- axis have their x-coordinates zero.

We can graph a line whose equation is given in any of the above mentioned form, by just algebraically transforming it into slope-intercept form.

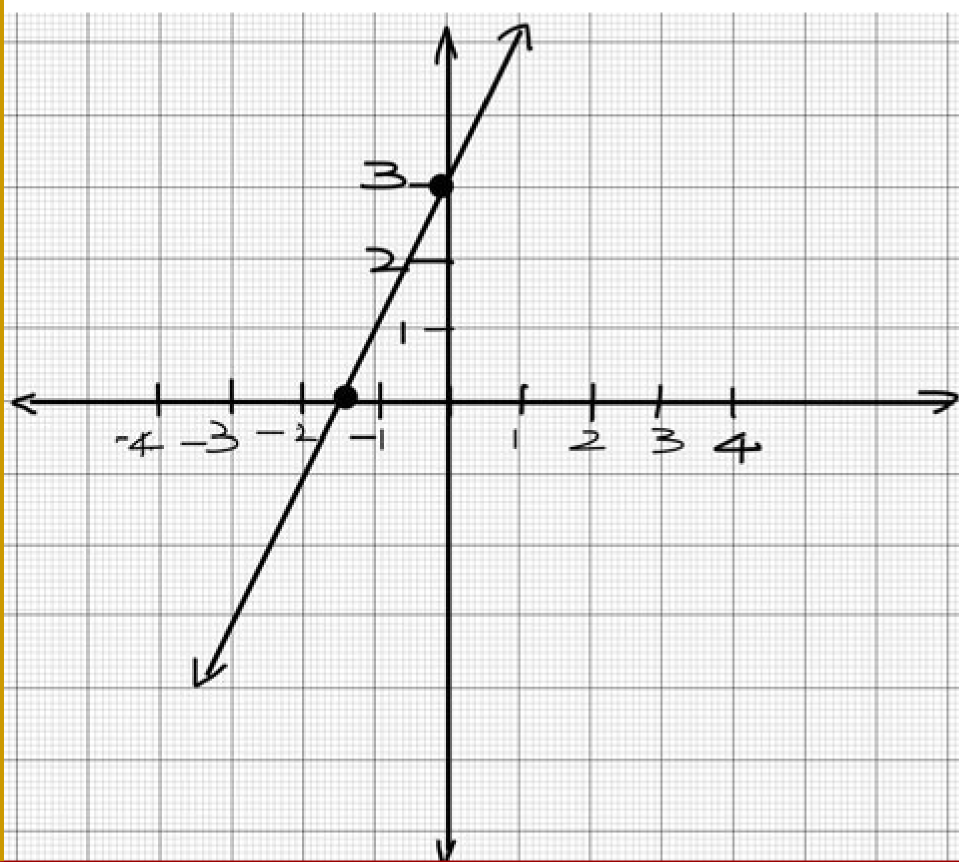
Example 5:

Graph the line $y - 5 = 2(x - 1)$

Solution: This equation is in point-slope form. First let us bring it to slope-intercept form:

$$\begin{aligned} y - 5 &= 2x - 2 \\ y &= 2x + 3 \end{aligned}$$

Now, we can graph this line following any one of the two methods shown in Example 1.
Alternately, we can take the first point as the given point $(1, 5)$ and take the slope of 2 to get the second point.

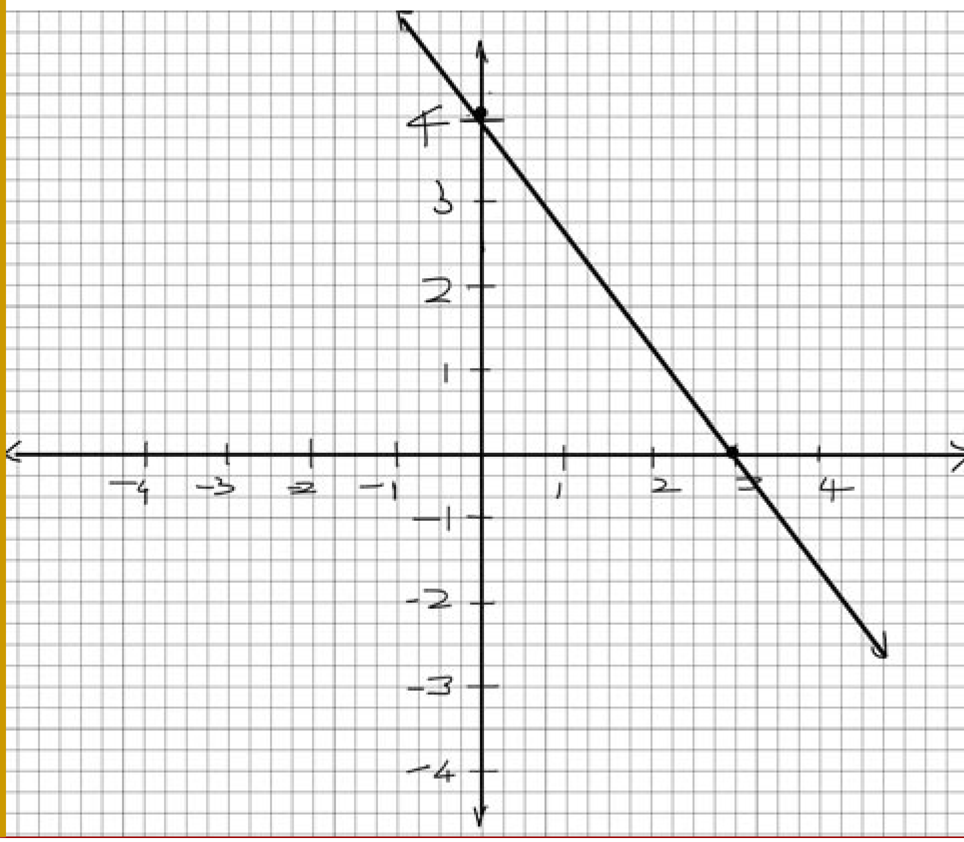


Example 6:

Graph the line $\frac{x}{3} + \frac{y}{4} = 1$

Solution: This equation is in two-intercepts form. So, the easiest way to graph this line is to plot the two intercepts

We note that the x- intercept is $(3, 0)$ and the y- intercept is $(0, 4)$.



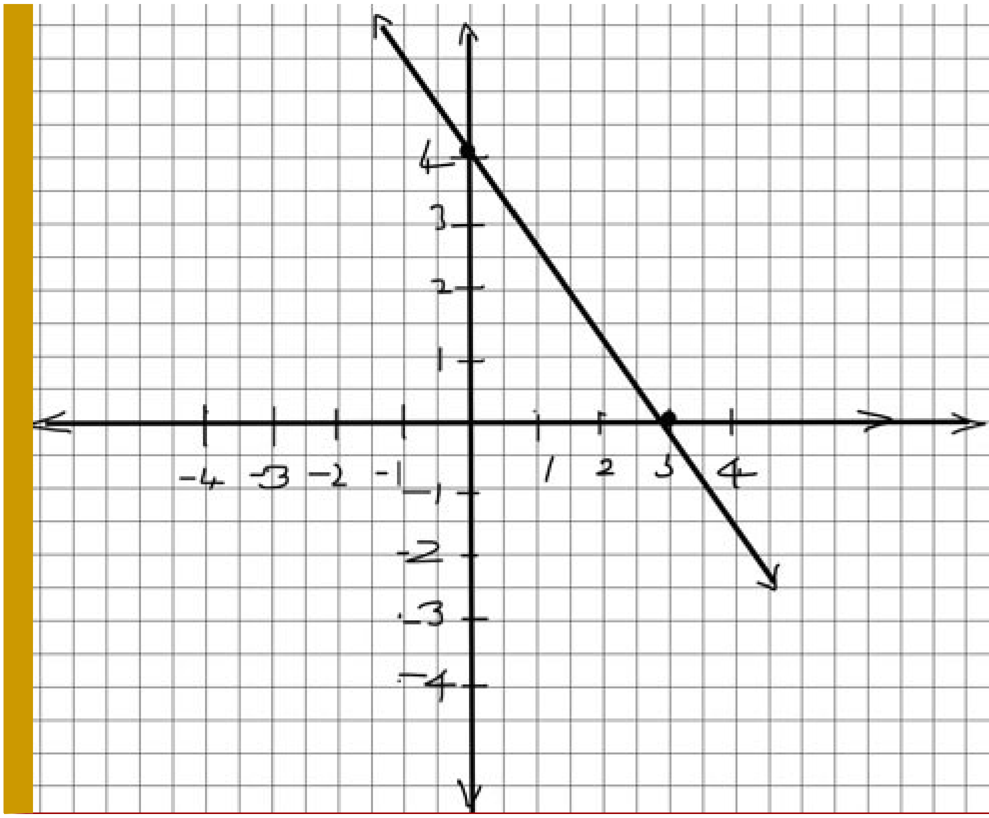
EXAMPLE 7:

Graph the line $4x + 3y = 12$.

Solution: This equation is in general form. We can bring this into slope-intercept form and then graph the line using any one of the methods stated in Example 2.

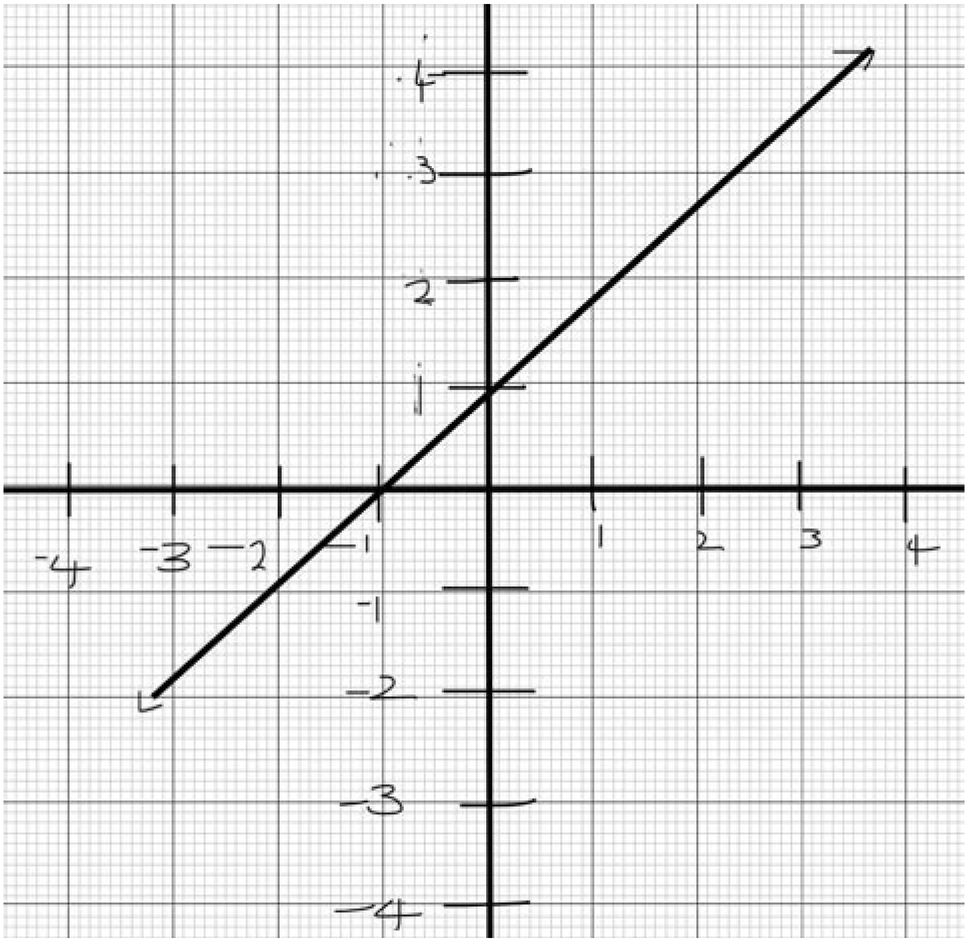
$$3y = -4x + 12$$
$$y = -\frac{4}{3}x + 4$$

So slope of this line is $-\frac{4}{3}$ and y-intercept is 4.

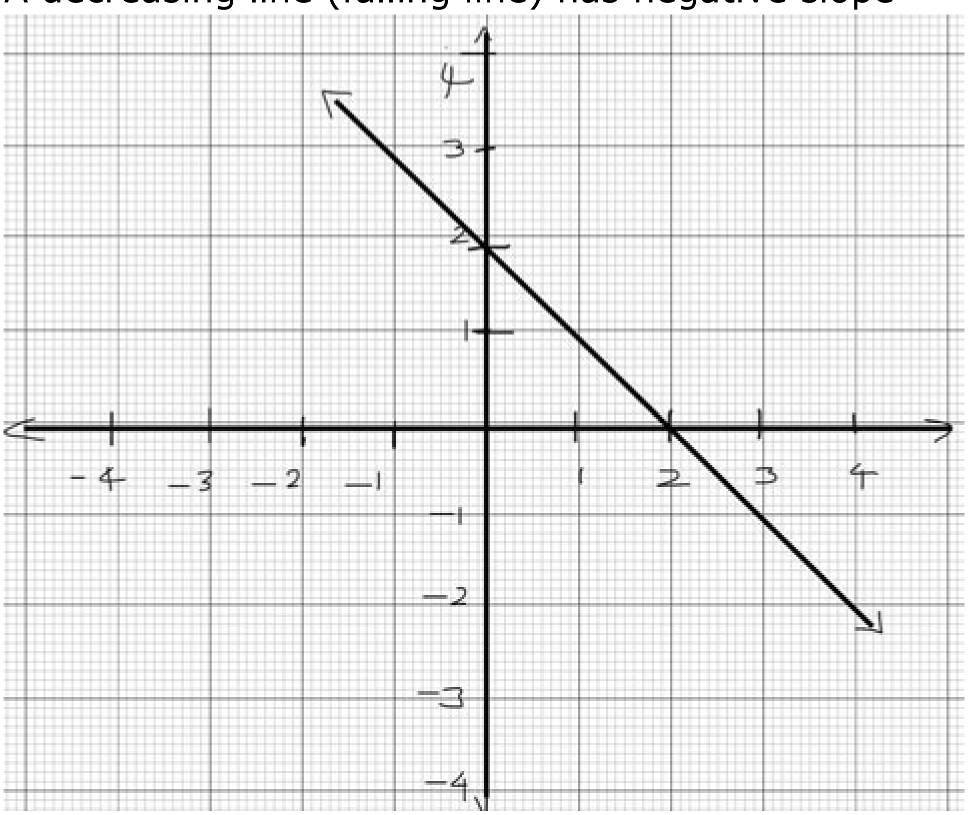


Following points are to be noted when working on problems involving lines and graphs:

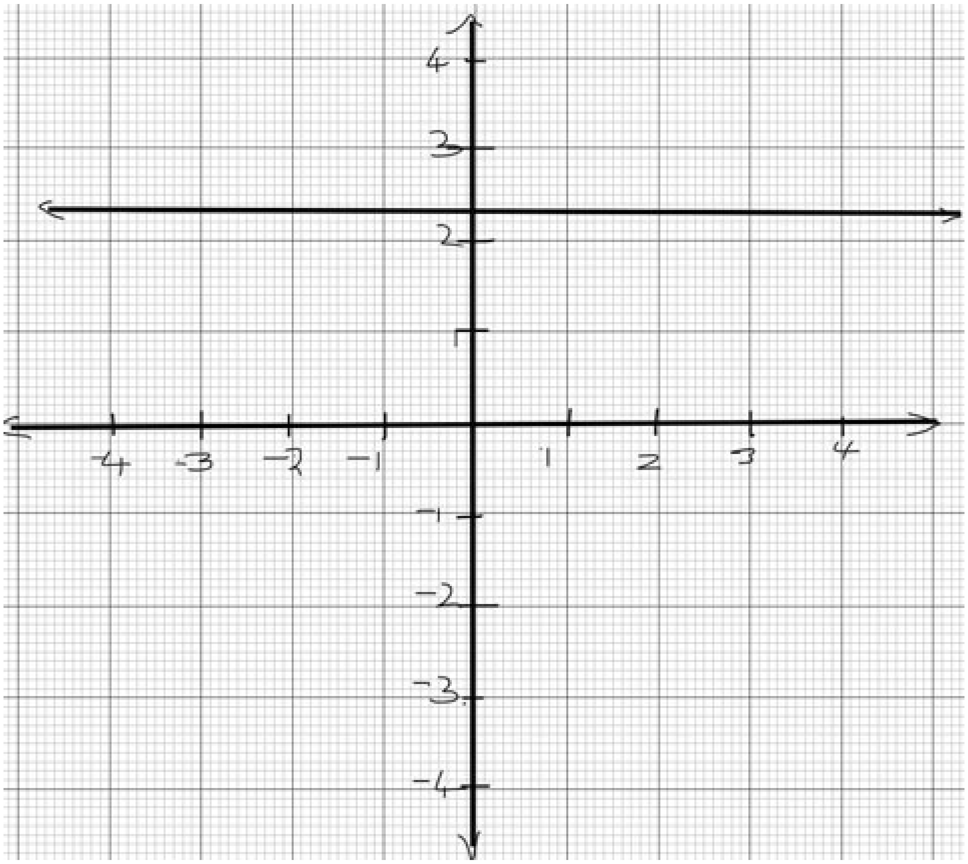
1. An increasing line (rising line) has positive slope.



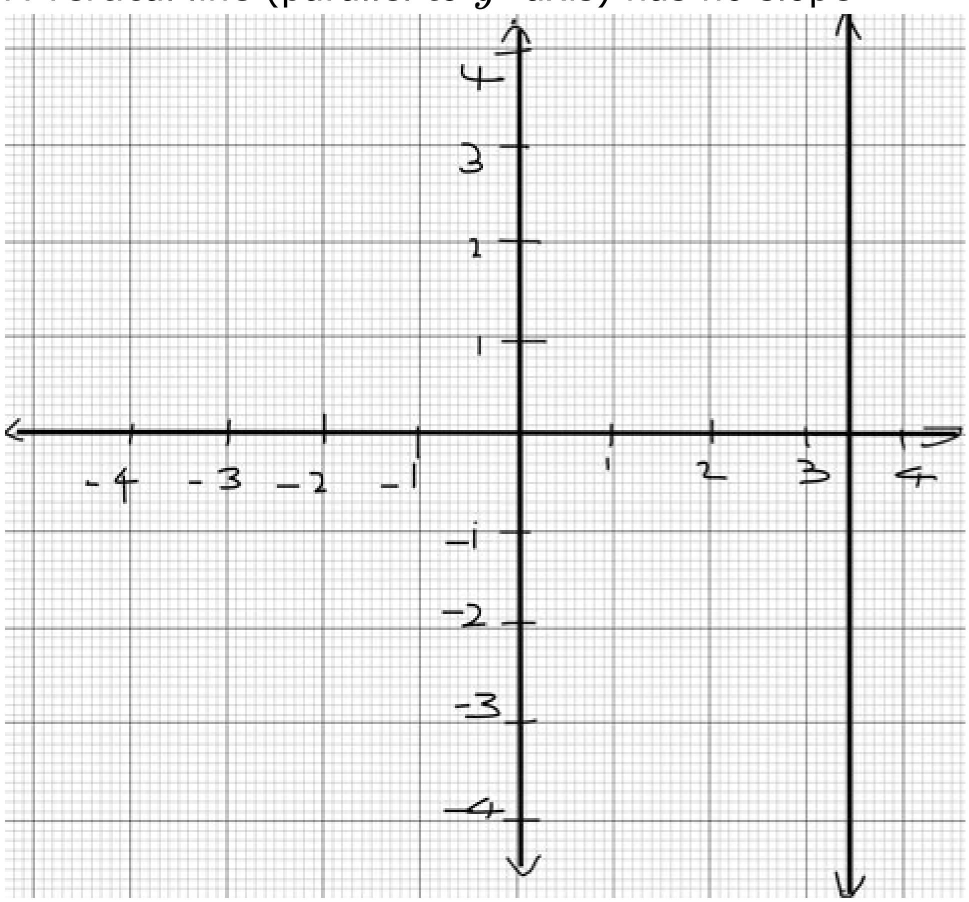
2. A decreasing line (falling line) has negative slope



3. A horizontal line (parallel to x- axis) has zero slope



4. A vertical line (parallel to y- axis) has no slope



5. Equation of a line passing through the origin has no constant term, as y-intercept is (0, 0)

If two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are parallel, then their slopes are equal. That is, $m_1 = m_2$. Conversely, if slopes of two lines are equal, these lines are parallel.

If two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are perpendicular, then their slopes are negative reciprocal of each other.

That is, $m_1 = -\frac{1}{m_2}$.

Conversely, if slopes of two lines are negative reciprocals of each other, then these lines perpendicular.

EXAMPLE 8:

Determine the equation of a line parallel to the line $y = 2x + 3$ that contains the points $(-1, 4)$.

Solution:
Equation of the given line is in slope- intercept form. So, its slope = 2.
Any line parallel to this line also has slope = 2.
Equation of the required line in point-slope form is

$$\begin{aligned} y - 4 &= 2(x + 1) \\ y - 4 &= 2x + 2 \\ y &= 2x + 6 \end{aligned}$$

EXAMPLE 9:

Find the equation of the line perpendicular to the line $3x - 4y = 8$ passing through $(3, -1)$.

Solution: Given line is in general form. So, we need to bring this into slope- intercept form first.

$$\begin{aligned} 3x - 4y &= 8 \\ -4y &= -3x + 8 \\ y &= \frac{3}{4}x + 2 \end{aligned}$$

Slope of the given line is $\frac{3}{4}$.
So, slope of a perpendicular line is $-\frac{4}{3}$.
Equation of the required line in point-slope form

$$\begin{aligned} y + 1 &= -\frac{4}{3}(x - 3) \\ y + 1 &= -\frac{4}{3}x + 4 \\ y &= -\frac{4}{3}x + 3 \end{aligned}$$

This is in slope-intercept form. We can write this in general form as follows:

$$\begin{aligned} y &= -\frac{4}{3}x + 3 \\ 3y &= -4x + 9 \\ 4x + 3y &= 9 \end{aligned}$$

Example 10:

Determine the equation of a line through a point A $(-2, 3)$ that is perpendicular to the line whose equation is $2x + y = 5$.

Solution:
Equation of the given line: $2x + y = 5$
Rewrite this in slope-intercept form:

$$y = -2x + 5$$

Slope of the given line is -2.
Slope of any line perpendicular to this line is $\frac{1}{2}$

The required line passes through A $(2, -3)$.
Equation of the required line in point-slope form is

$$y + 3 = \frac{1}{2}(x - 2)$$

Simplifying and writing the equation in general form:

$$\begin{aligned} 2y + 6 &= x - 2 \\ 2y - x &= -8 \\ x - 2y &= 8 \end{aligned}$$

Example 11:

Find the equation, in general form, of a line through the point $(-5, 4)$ which is perpendicular to the line $3x - 4y = 7$

Solution: Equation of the given line in slope-intercept form:

$$\begin{aligned} 4y &= 3x - 7 \\ y &= \frac{3}{4}x - \frac{7}{4} \end{aligned}$$

Slope of the given line is $\frac{3}{4}$. So, slope of any line perpendicular to this line is $-\frac{4}{3}$.
This required line passes through the point $(-5, 4)$. So, the required equation in point-slope form:

$$y - 4 = -\frac{4}{3}(x + 4)$$

General form of the equation:

$$\begin{aligned} 3y - 12 &= -4x - 12 \\ 4x + 3y &= 0 \end{aligned}$$

Practice Problems

(1) Find the distance between the points $(1, 4)$ and $(6, 2)$

(2) Find the distance between the points $(-4, -3)$ and $(2, 1)$

(3) Find the midpoint of the line segment joining the points $(-1, 3)$ and $(4, 7)$

(4) Find the midpoint of the line segment whose end points are $(5, -2)$ and $(-3, -4)$

(5) Find the area and perimeter of the figure determined by the points A$(-3, -1)$, B$(-1, 3)$, C$(7, 3)$ and D$(5, -1)$

(6) Find the equation of the line passing through the point $(-3, -4)$ and has a slospe of 2

(7) Write the equation of a line with slope -2 and y- intercept (0, 1)

(8) Write the equation of a line with slope 2 that passes through the point $(-1, 4)$

(9) Find the equation of a line in slope-intercept form, what passes through the points $(4, 1)$ and $(-2, -1)$

(10) Find the general form of the equation of am line that passes through the points $(1, -3)$ and $(-4, 2)$.

(11) Graph the line whose equation is given by $2x + y = 4$

(12) Find the value of x so that the line with slope 2 passes through the points $(5, 9)$ and $(x, 3)$

(13) Find the value of k so that the line with equation $5x + ky = 3$ passes through the point $(2, -1)$

(14) Find the values of a and b so that the points $(a, 14)$ and $(18, b)$ lie on the graph of the line $3x - 2y = 14$

(15) Find the equation of a line passing through the point $(1, 2)$ and parallel to the line whose equation is $y = -2x + 4$

(16) Find the equation, in general form, of a line passing through the point $(-1, 3)$ and perpendicular to the line whose equation is $2x + 3y = 6$

(17) Find the value of a if the slope of a line passing through the points $(-6, 13)$ and $(3, a)$ is $-\frac{1}{3}$

(18) A triangle has vertices A$(1, -1)$, B$(-1, 3)$ and C$(2, 2)$. Find the slope and equation in slope-intercept form of the median of the triangle through C

(19) A line joining A $(-2, 3)$ and B$(3, -5)$ is parallel to the line joining C $(0, -4)$ and D $(-3, k)$. Find the value of k

(20) A line joining A $(0, -4)$ and B $(3, a)$ is perpendicular to the line joining C $(-1, -3)$ and D $(-4, 6)$. Find the value of a

(21) Points A $(0, 3)$, B $(-1, 2)$ and C $(-1, 1)$ are vertices of a triangle. find the slope of the altitude of side AB.

(22) Determine whether the points A $(-2, 3)$, B$(8, 5)$, and C $(3, 4)$ are collinear.