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PRECALCULUS

Chapter P - Preliminary Concepts

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P2. Linear Equations and Inequalities

An equation is a mathematical statement with equality between two expressions. A linear equation is an equation where no higher powers of the variable are involved on either side of the equation. Each term of the equation should be either of degree one or a constant.

Let u, v, w and z be real numbers, variables, or algebraic expressions. Then, the following properties would be useful in solving equations.

1. Reflexive property: $u = u$
2. Symmetric property: $u + v = v + u$
3. Transitive property: If $u = v$ and $v = w$, then $u = w$
4. Addition property: If $u = v$ and $w = z$, then $u + w = v + z$
5. Multiplication property: If $u = v$ and $w = z$, then $uw = vz$

A solution to an equation in x is a value of x for which the equation is true.

This means that, when you substitute that value for x, the left hand side and the right hand side expressions become equal. A solution to an equation is also called a root or a zero of the equation.

EXAMPLE 1:

Prove that $x = -2$ is a solution of the equation $x^2 - 3x + 6 = 0$.

Solution: Substituting $x = -2$ in the equation gives $(-2)^2 - 3(-2) + 6 \neq 0$. So, $x = -2$ is not a solution to this equation.

A linear equation in one variable x can be written in the form $ax + b = 0$ where a and b are real numbers with $a \neq 0$.

Two or more equations are called EQUIVALENT EQUATIONS if they have the same solutions. Following are operations that produce equivalent equations:

1. Perform the same operation on both sides. That is, you can add, subtract, multiply, or divide both sides by the same real number.
2. Combine like terms, reduce fractions, and remove grouping symbols

Example 2:

For the equation $x + 3 = 7$, subtracting 3 from both sides would result in an equivalent equation $x = 4$.

Example 3:

Solve the equation $5x = 2x + 4$

Solution: Subtract $2x$ from both sides gives $3x = 4$. Divide both sides by 3 gives $x = \frac{4}{3}$. So, solution to this equation is $x = \frac{4}{3}$.

Example 4:

Solve the equation $2(2x - 3) + 3(x + 1) = 5x + 2$

Solution:

$$\begin{aligned} 2(2x-3)+3(x+1) &= 5x+2 \\ 4x-6+3x+3 &= 5x+2 \\ 7x-3 &= 5x+2 \\ 2x &= 5 \\ x &= 2.5 \end{aligned}$$

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EXAMPLE 5:

Solve the equation $3(2y-3)=6(y+1)-15$

Solution:

$$\begin{aligned} 3(2y-3)=6(y+1)-15 \\ 6y-9=6y+6-15 \\ 6y-9=6y-9 \end{aligned}$$

.

Since the given equation is equivalent to $6y-9=6y-9$ which is true for all value of y, this equation is called an identity.

The solution set of such equations is the set of all real numbers..

EXAMPLE 6:

Solve the equation $3(2x-3)=6(x+1)-10$

Solution:

$$\begin{aligned} 3(2x-3)=6(x+1)-10 \\ 6x-9=6x+6-10 \\ 6x-9=6x-4 \\ -9=-4 \\ 0=5 \end{aligned}$$

Since the given equation is equivalent to a FALSE STATEMENT $0=5$, this equation is called a contradictory equation. It has no solution.

If an equation involves fractions, first find the least common denominator (LCD) of all terms on both sides and then multiply both sides of the equation by the LCD.

Example 7:

Solve the equation $\frac{5y-2}{8} = 2 + \frac{y}{4}$

Solution:

Here,we have denominators 8, 1,and 4.
The LCD is 8. So, multiply each term on both sides by 8.

$$\begin{aligned} \frac{5y-2}{8} &= 2 + \frac{y}{4} \\ 8 \times \frac{5y-2}{8} &= 8 \times 2 + 8 \times \frac{y}{4} \\ 5y - 2 &= 16 + 2y \\ 3y &= 18 \\ y &= 6 \end{aligned}$$

A linear inequality in x is one that can be written in any one of the following forms:

$$ax + b < 0$$

$$ax + b \leq 0$$

$$ax + b > 0$$

$$ax + b \geq 0$$

Solving an inequality in x means finding all values of x for which the inequality is true. The set of all such values is called the solution set of the inequality. Following are the properties we use to solve inequalities:

Let u, v, w, and z be real numbers, variables, or algebraic expressions and let c be a real number. Then,

1. Transitive property: If $u < v$ and $v < w$, then $u < w$
2. Addition property: If $u < v$, then $u + w < v + w$. If $u < v$ and $w < z$, then $u + w < v + z$
3. Multiplication property: If $u < v$ and $c > 0$, then $uc < vc$. If $u < v$ and $c < 0$, then $uc > vc$
4. Comparison property: Let a, b,and c be any real numbers. Then, exactly one of the following statements is true:
   $a < b, \quad a = b, \quad or \quad a > b$

We solve an inequality by making it into EQUIVALENT inequalities by performing one or more of the folloiwng transformations:

1. Simplifying either side of the inequality.
2. Adding (or subtracting) from each side of the inequality the same number or expression.
3. Multiplying (or dividing) each side of the inequality by the same POSITIVE number.
4. Multiplying (or dividing) each side of the inequality by thesame NEGATIVE number and REVERSING the direction of the inquality.

The solution set of an inequality is written as an interval and as an inequality, and also should be shown on a number line.

Example 8:

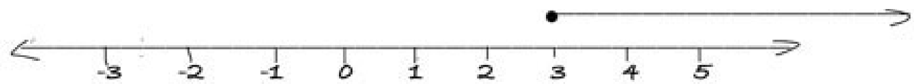
Solve the inequality $2(5-3x)+3(2x-1) \leq 2x+1$

Solution:

$$\begin{aligned} 2(5-3x)+3(2x-1) &\leq 2x+1 \\ 10-6x+6x-3 &\leq 2x+1 \\ 7 &\leq 2x+1 \\ 6 &\leq 2x \\ 3 &\leq x \end{aligned}$$

So, the solution for this inequality is $x \geq 3$. This solution is written as

$[3, \infty)$ as an interval and $\{x/x \in R, x > 3\}$. The solution is also represented on the number line as



EXAMPLE 9:

Solve the inequality $5(3-x) < 7-x$

Solution:

$$\begin{aligned} 5((3-x) &< 7-x \\ 15-5x &< 7-x \\ -4x &< -8 \\ x &> 2 \end{aligned}$$

So, the solution for this inequality consists of all real numbers greater than 2. This solution is written as $(2, \infty)$ and as $\{x/x \in \mathcal{R}, x > 2\}$

EXAMPLE 10:

Solve the inequality $4x \geq 2(3+2x)$

Solution:

$$\begin{aligned} 4x &\geq 2(3+2x) \\ 4x &\geq 6+4x \\ 0 &\geq 6 \end{aligned}$$

Since the equivalent inequality $0 \geq 6$ is FALSE, the given inequality is FALSE and has no solution. The solution set for this false inequality is ϕ .

A combined or compound inequality is where two inequalities are combined. Procedures of solving combined inequalities are basically two methods:

1. Conjunction: These are applicable when two inequalities are combined with "and" and the solution set would make BOTH statements true.
2. Disjunction: These are applicable when two inequalities are combined with "or" and the solution set would make AT LEAST ONE of the statements true.

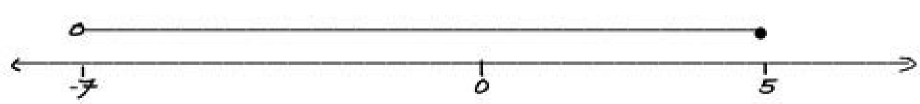
Example 11:

Solve the inequality $-3 < \frac{2x+5}{3} \leq 5$

Solution:

$$\begin{array}{rcll} -3 & < & \frac{2x+5}{3} & \leq & 5 \\ -9 & < & 2x+5 & \leq & 15 \\ -14 & < & 2x & \leq & 10 \\ -7 & < & x & \leq & 5 \end{array}$$

The solution set is all real numbers greater than - 7 and less than or equal to 5. Since the solution set satisfies both the inequalities, this is an "and" statement. This is written as (- 7, 5] in interval form and $\{x/x \in \mathcal{R}, -7 < x \leq 5\}$.



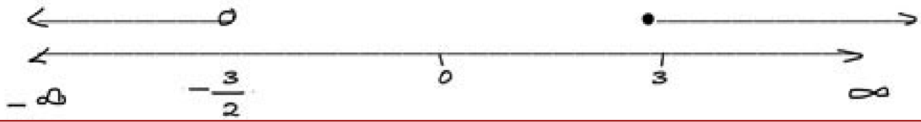
Example 12:

Solve the inequality $7 - 2x \leq 1$ or $3x + 10 < 4 - x$

Solution:

$$\begin{array}{rcllrcl} 7-2x & \leq & 1 & \quad or \quad & 3x+10 & < & 4-x \\ -2x & \leq & -6 & \quad or \quad & 4x & < & -6 \\ x & \geq & 3 & \quad or \quad & x & < & -\frac{3}{2} \end{array}$$

The solution set is all real numbers less than $-\frac{3}{2}$ OR greater than or equal to 3 .The solution set satisfies at least one of the inequalities. This is an "or" statement. This is written as $(-\infty, -\frac{3}{2}) \cup [3, \infty)$ in interval form and $\{x/x \in \mathcal{R}, x < -\frac{3}{2} \quad or \quad x \geq 3\}$ in set notation.



Practice Problems

Solve the equations and write the solutions as a set. Solve the inequalities and write the solutions as an inequality, as an interval, and also show it on a number line

(1) $4 - 3y = 2(y + 4)$

(2) $\frac{2x}{4} = \frac{4}{5}$

(3) $2(3 - 4z) - 5(2z + 3) = z - 17$

(4) $\frac{2x-3}{4} + 5 = 3x$

(5) $\frac{t-1}{3} + \frac{t+5}{4} = \frac{1}{2}$

(6) $2x - 1 \leq 4x + 3$

(7) $-1 \leq 3x - 2 < 7$

(8) $2(5 - 3x) + 3(2x - 1) \leq 2x + 1$

(9) $\frac{5x+8}{4} \leq -3$

(10) $\frac{3x-2}{5} > -1$

(11) $1 > \frac{3y-1}{4} > -1$

(12) $-6 < 5t - 1 < 0$

(13) $\frac{3-x}{2}+\frac{5x-2}{3}<-1$

(14) $\frac{3-4y}{6}-\frac{2y-3}{8}\geq 2-y$

(15) $\frac{1}{2}(x+3)+2(x-4)<\frac{1}{3}x$

(16) $4\geq\frac{2y-5}{3}\geq -2$

(17) $4x-7>x-4$

(18) $5(2u+3)>2(u-3)$

(19) $-5<3x-2<0$

(20) $1+2x\leq 2(x-1)$

(21) $\frac{2y-3}{2}+\frac{3y-1}{5}<y-1$

(22) $k-3(2-4k)<7-(8k-9+k)$

(23) $4(y+2)-9y>y-3(2y+1)-1$

(24) $-1>2t-5>-9$

(25) $2z-1\leq 5 \quad or \quad 3z-5>10$

(26) $-1\leq 3z+2\leq 8$

(27) $2t+7\geq 13 \quad or \quad 5t-4<6$

(28) $x-7<3x-5<x+11$

(29) $2x+3>1 \quad and \quad 5x-9\leq 6$